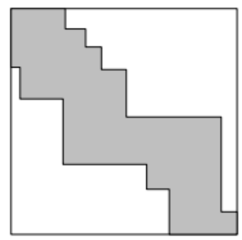
NYSML
Team Round
20 minutes -- no calculators permitted
5 points each -- 50 total points

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

T1. The figure below shows a shaded polygon P inscribed in a square. The polygon P has each of its sides parallel to one of the sides of the square.



Given that P has perimeter 36 units and area 30 square units, compute the fraction of the area of the square that P covers.

T2. Mr. Pythagoras and Ms. Germain each have five students who took a math test. All ten students received distinct integer grades. Four summary statistics for each teacher are shown in the chart below.

Teacher	Minimum	Maximum	Median	Mean
Mr. Pythagoras	70	96	83	84
Ms. Germain	55	72	68	64

Compute the median of the set containing all ten students' grades.

T3. A globe has longitudinal lines spaced out every 15 degrees and latitudinal lines spaced out every 10 degrees until the north and south poles, which are labeled on the globe as latitudinal points. Compute the number of distinct points of intersection on the globe between the longitudinal lines and the latitudinal lines and points.

T4. Compute the least positive integer that does not divide $26!$ and also is not divisible by any prime greater than 26.

T5. Let S be the set of real values of x that satisfy $x^4 - ax^3 - ax^2 + a^2x = 0$ for some constant a. Let A be the set of all real values of a such that S has at least two elements that differ by 12. Compute A.

T6. Compute the number of eight-digit positive integers that have no contiguous substring divisible by 9.

T7. Compute the real number b such that $\log_b(2b^3 - 4b^2 + 5b - 2) = \log_b(2b^2 - 1)$.

T8. After expanding $(x + 3y + 2z)^9$ and collecting like terms, some of the terms are expressible as $n^2x^ay^bz^c$, where a, b, and c are nonnegative integers and n is a positive integer. Compute the sum of all possible values of n.

T9. A right pyramid with height 84 has a rectangular base with length 26 and width 70. A cut parallel to the base of the pyramid splits it into a smaller pyramid with volume 3185 and a frustum. Compute the surface area of the frustum.

T10. In the below number puzzle, the dark outlines define different regions, where each cell in the same region has the same nonzero digit, and no two such regions have the same digit. Each row and column define a three-digit number, with the following constraints.

Row 1: A multiple of 31

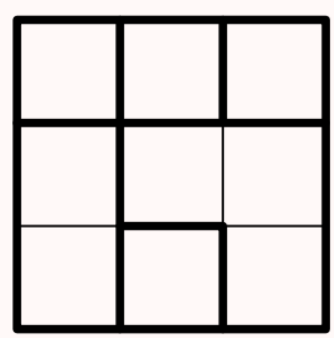
Row 2: Can be written as the sum of two 3-digit palindromes

Row 3: A perfect square

Column 1: Can be written as the sum of two squares

Column 2: Can be written as the product of two 2-digit integers

Column 3: Is a multiple of a 1-digit prime

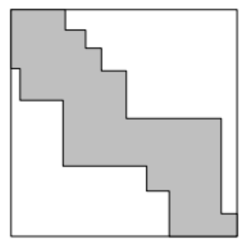


Compute the sum of the three 3-digit numbers in the three rows.

NYSML
Team Round
20 minutes -- no calculators permitted
5 points each -- 50 total points

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

T1. The figure below shows a shaded polygon P inscribed in a square. The polygon P has each of its sides parallel to one of the sides of the square.



Given that P has perimeter 36 units and area 30 square units, compute the fraction of the area of the square that P covers.

T1-Sol. $\boxed{\frac{10}{27}}$ Consider all of the vertical sides of P on the left of P. Shifting them to the left causes them to coincide with the left side of the square with no overlap, so the total length of these vertical sides is one side length of the square. Similarly, the right-most vertical sides of P, the top-most sides of P, and the bottom-most sides of P each have total length equal to the side length of the square. Put informally, one can "unfold" the perimeter of P to make it the perimeter of the square. Thus P and the square have the same perimeter, given to be 36 units in the problem statement. This implies that the square has side length $\frac{1}{4} \cdot 36 = 9$ units, and hence it has area $9^2 = 81$ square units. The polygon P therefore covers a fraction of $\frac{30}{81} = \mathbf{\frac{10}{27}}$ of the area of the full square.

T2. Mr. Pythagoras and Ms. Germain each have five students who took a math test. All ten students received distinct integer grades. Four summary statistics for each teacher are shown in the chart below.

Teacher	Minimum	Maximum	Median	Mean
Mr. Pythagoras	70	96	83	84
Ms. Germain	55	72	68	64

Compute the median of the set containing all ten students' grades.

T2-Sol. $\boxed{71}$ The median is the average of the fifth and sixth grades when sorted from least to greatest. Three of Ms. Germain's students' grades are less than the lowest grade among

Mr. Pythagoras's students. The second highest grade of Ms. Germain's students could be either 69 or 71. If it was 71, then the other grade can be calculated using the average: $(64 \times 5) - (55 + 68 + 71 + 72) = 54$, which is lower than the minimum and thus a contradiction. Therefore the second highest grade for Ms. Germain's students is 69 and is lower than the lowest score of Mr. Pythagoras, which makes the fifth lowest grade a 70.
The sixth lowest grade can be either 72 from Ms. Germain's class or 71 from Mr. Pythagoras's class. If the grade is 71, then the other grade for Mr. Pythagoras's students is $(84 \times 5) - (70 + 71 + 83 + 96) = 100$, which is greater than the maximum and thus a contradiction. Therefore the sixth lowest grade is 72. The median grade of all ten students is then $\frac{70+72}{2} = \mathbf{71}$.
Note that the conditions of the problem can be satisfied if Mr. Pythagoras's students score 70, 76, 83, 95, and 96 and Ms. Germain's students score 55, 56, 68, 69, and 72.

T3. A globe has longitudinal lines spaced out every 15 degrees and latitudinal lines spaced out every 10 degrees until the north and south poles, which are labeled on the globe as latitudinal points. Compute the number of distinct points of intersection on the globe between the longitudinal lines and the latitudinal lines and points.

T3-Sol. $\boxed{410}$ In total, there are 360 degrees of longitude, so there will be $\frac{360}{15} = 24$ lines of longitude. There are 8 lines of latitude above the equator and 8 lines of latitude below the equator for a total of $(2 \times 8) + 1 = 17$ latitudinal lines. Each longitudinal line crosses each latitudinal line exactly once at different points on the globe, which is $24 \times 17 = 408$ intersections. Each longitudinal line starts from one pole and goes across the globe to the other pole, so all longitudinal lines intersect the two poles at the same two points on the globe, which account for 2 additional intersections. The total number of intersections on the globe is $(24 \times 17) + 2 = \mathbf{410}$.

T4. Compute the least positive integer that does not divide 26! and also is not divisible by any prime greater than 26.

T4-Sol. $\boxed{289}$ Using Legendre's factorial formula, the number 26! has the prime factorization $26! = 2^{23} \cdot 3^{10} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^1 \cdot 19^1 \cdot 23^1$. The desired number is not divisible by any prime greater than 26, so it may be written as $2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot 11^{a_5} \cdot 13^{a_6} \cdot 17^{a_7} \cdot 19^{a_8} \cdot 23^{a_9}$, where at least one a_i is greater than the corresponding exponent in 26!.
To minimize the number, set one of the nine a_i to be 1 greater than the corresponding exponent and all the other a_i to 0. This creates nine possible solutions: $2^{24}, 3^{11}, 5^7, 7^4, 11^3, 13^3, 17^2, 19^2, 23^2$. The least of these is $17^2 = \mathbf{289}$.

T5. Let S be the set of real values of x that satisfy $x^4 - ax^3 - ax^2 + a^2x = 0$ for some constant a. Let A be the set of all real values of a such that S has at least two elements that differ by 12.

Compute A.

T5-Sol. $\boxed{\{-12, 9, 12, 16, 36, 144\}}$ Factor to obtain
$x^4 - ax^3 - ax^2 + a^2x = x^3(x-a) - ax(x-a) = x(x-a)(x^2-a)$. This implies $S = \{0, a, \pm\sqrt{a}\}$, and these elements do not all exist unless $a \geq 0$. Note which elements can differ by 12.
If $a - 0 = 12$, then $a = 12$.
If $0 - a = -12$, then $a = -12$, in which case S has only two elements.
If $\sqrt{a} - 0 = 12$, then $a = 144$.
If $0 - (-\sqrt{a}) = 12$, then $a = 144$.
If $a - \sqrt{a} = 12$, then $(\sqrt{a})^2 - \sqrt{a} - 12 = 0$, which implies $(\sqrt{a} - 4)(\sqrt{a} + 3) = 0$, which implies $a = 16$.
If $a - (-\sqrt{a}) = 12$, then $(\sqrt{a})^2 + \sqrt{a} - 12 = 0$, which implies $(\sqrt{a} + 4)(\sqrt{a} - 3) = 0$, which implies $a = 9$.
The equations $\pm\sqrt{a} - a = 12$ have no real solutions.
If $\sqrt{a} - (-\sqrt{a}) = 12$, then $\sqrt{a} = 6$ and so $a = 36$. The answer is $A = \mathbf{\{-12, 9, 12, 16, 36, 144\}}$.

T6. Compute the number of eight-digit positive integers that have no contiguous substring divisible by 9.

T6-Sol. $\boxed{40320}$ Note that the problem statement implies that there are no 0s or 9s in the number. Look at the partial sums of the digits; the claim is that they must all be distinct modulo 9. If not, then one could take the difference between two partial sums that are the same modulo 9, which would produce a contiguous substring whose digits sum to a multiple of 9, and that would be a contradiction. Thus, of the 8 partial sums, choose them to be $1, 2, \ldots, 8 \mod 9$, in some order. Moreover, for any such choice, there is a unique eight-digit number that works, because one can choose the kth digit to make the kth partial sum equal to the desired number mod 9. Thus the answer is $8! = \mathbf{40320}$.

T7. Compute the real number b such that $\log_b(2b^3 - 4b^2 + 5b - 2) = \log_b(2b^2 - 1)$.

T7-Sol. $\boxed{\frac{2+\sqrt{2}}{2} \text{ or } 1 + \frac{\sqrt{2}}{2}}$ Let $f(b) = 2b^3 - 4b^2 + 5b - 2$ and $g(b) = 2b^2 - 1$. For such a b to exist, two conditions must be satisfied:

1. $f(b) = g(b) > 0$,
2. $b > 0, b \neq 1$.

To find any such b, solve for $f(b) - g(b) = 0$, which is equivalent to $2b^3 - 6b^2 + 5b - 1 = 0$. Then, either by inspection or by the Rational Root Theorem, one of the roots is $b = 1$. This is not a viable solution because the base of a logarithm cannot be 1. The cubic can now be factored into a linear term and quadratic $Q(b)$, and so $2b^3 - 6b^2 + 5b - 1 = 0 \implies (b-1)(2b^2 - 4b + 1) = 0$.

Focusing on $Q(b) = 2b^2 - 4b + 1$, its two roots are $b_- = \frac{2-\sqrt{2}}{2}$ and $b_+ = \frac{2+\sqrt{2}}{2}$ by the quadratic formula. Both roots must satisfy $f(b) > 0$ or $g(b) > 0$ because the argument of a logarithm must be positive. Using the fact that $Q(b_-) = Q(b_+) = 0$ helps to determine whether $g(b_+) > 0$ and $g(b_-) > 0$, as shown below.

$$g(b_-) = 2b_-^2 - 1 = (2b_-^2 - 4b_- + 1) + 4b_- - 2 = 4b_- - 2 = 2\left(1 - \sqrt{2}\right)$$

$$g(b_+) = 2b_+^2 - 1 = (2b_+^2 - 4b_+ + 1) + 4b_+ - 2 = 4b_+ - 2 = 2\left(1 + \sqrt{2}\right)$$

With $g(b_-) < 0$ and $g(b_+) > 0$, the only viable solution is $b = \mathbf{\frac{2+\sqrt{2}}{2}}$.

T8. After expanding $(x + 3y + 2z)^9$ and collecting like terms, some of the terms are expressible as $n^2 x^a y^b z^c$, where a, b, and c are nonnegative integers and n is a positive integer. Compute the sum of all possible values of n.

T8-Sol. $\boxed{322}$ From the Multinomial Theorem, the expansion of this product is

$$\sum_{\substack{a,b,c\geq 0\\ a+b+c=9}} \binom{9}{a,b,c}(x)^a(3y)^b(2z)^c.$$

Therefore each coefficient is of the form $\binom{9}{a,b,c} \cdot 3^b \cdot 2^c$ where a, b, and c are nonnegative integers that sum to 9. The problem requests to compute the number of distinct perfect squares that can be expressed in this way.

The multinomial coefficient is equal to $\binom{9}{a,b,c} = \frac{9!}{a!\cdot b!\cdot c!}$. Notice that the numerator is divisible by 7 and not 7^2, while the denominator is only divisible by 7 if one of a, b, or c is at least 7. As a result, the expression

$$\binom{9}{a,b,c} \cdot 3^b \cdot 2^c$$

is divisible by 7 and not 7^2 whenever $a, b, c < 7$. This forces the coefficient to not be a perfect square, so the only perfect square coefficients will have at least one of a, b, or c to be at least 7. Therefore the only triples (a, b, c) left to consider are permutations of the triples $(7, 1, 1)$, $(7, 2, 0)$, $(8, 1, 0)$, and $(9, 0, 0)$.

The multinomial coefficients for each of these triples are:

$$\binom{9}{7,1,1} = 2^3 \cdot 3^2, \quad \binom{9}{7,2,0} = 2^2 \cdot 3^2, \quad \binom{9}{8,1,0} = 3^2, \quad \binom{9}{9,0,0} = 1.$$

Therefore if (a, b, c) is $(7, 1, 1)$ in some order, the way $\binom{9}{7,1,1} \cdot 3^b \cdot 2^c$ is a perfect square is if b is odd and c is even, which is impossible since $(7, 1, 1)$ are all odd. If $(a, b, c) = (7, 2, 0)$ in some order, then if $\binom{9}{7,2,0} \cdot 3^b \cdot 2^c$ is a perfect square then b and c are both even, which yields the two ordered triples $(a, b, c) = (7, 2, 0)$ and $(a, b, c) = (7, 0, 2)$, giving coefficients 324 and 144, respectively. If $(a, b, c) = (8, 1, 0)$ in some order, then b and c must again be even, so $(a, b, c) = (1, 8, 0)$ or $(a, b, c) = (1, 0, 8)$, yielding $9 \cdot 2^8$ and $9 \cdot 3^8$ as coefficients. Finally, if $(a, b, c) = (9, 0, 0)$ in some order, then b and c must both be even, and consequently the corresponding coefficient is 1. In conclusion, the perfect square coefficients are given by

$$1, \quad 144, \quad 324, \quad 9 \cdot 2^8, \quad 9 \cdot 3^8.$$

Their square roots are, respectively, 1, 12, 18, 48, and 243, and the sum of these values is **322**.

T9. A right pyramid with height 84 has a rectangular base with length 26 and width 70. A cut parallel to the base of the pyramid splits it into a smaller pyramid with volume 3185 and a frustum. Compute the surface area of the frustum.

T9-Sol. $\boxed{10136 - 812\sqrt[3]{2} \text{ or } 28(362 - 29\sqrt[3]{2})}$ Let the length and width of the base of the bigger pyramid be $L = 26$ and $W = 70$. Similarly, let the height of the bigger pyramid be $H = 84$ and the volume of the smaller pyramid be $V = 3185$.
The frustum has six faces, two bases and four lateral faces. The area of the larger base face is simply $LW = 1820$. To compute the area of the remaining faces, first calculate the volume, V_B and the lateral surface area, A_B, of the bigger pyramid. Both pyramids are similar, so the ratio of their surface areas is equal to the $\frac{2}{3}$ power of the ratio of their volumes. This implies

$$\frac{A_S}{A_B} = \sqrt[3]{\left(\frac{V_S}{V_B}\right)^2} = \sqrt[3]{\left(\frac{V}{V_B}\right)^2}.$$

The volume and lateral surface area of the large pyramid is below.

$$V_B = \frac{1}{3}LWH = 50960$$

$$\begin{aligned} A_B &= W\sqrt{\left(\frac{L}{2}\right)^2 + H^2} + L\sqrt{\left(\frac{W}{2}\right)^2 + H^2} \\ &= 70\sqrt{13^2 + 84^2} + 26\sqrt{35^2 + 84^2} \\ &= 70\sqrt{7225} + 26\sqrt{7^2(5^2 + 12^2)} \\ &= 70 \cdot 85 + 26 \cdot 7 \cdot 13 \\ &= 8316 \end{aligned}$$

Then the lateral surface area of the frustum is calculated below by scaling A_B down using both volumes.

$$
\begin{aligned}
A_F &= A_B - A_S \\
&= A_B\left(1 - \sqrt[3]{\left(\frac{V}{V_B}\right)^2}\right) \\
&= 8316\left(1 - \sqrt[3]{\left(\frac{3185}{50960}\right)^2}\right) \\
&= 8316\left(1 - \sqrt[3]{\left(\frac{1}{16}\right)^2}\right) \\
&= 8316\left(1 - \frac{\sqrt[3]{2}}{8}\right)
\end{aligned}
$$

Using the same scaling factor, the smaller base of the frustum is $LW \cdot \sqrt[3]{\left(\frac{V}{V_B}\right)^2}$. The final calculation for the surface area A_F of the frustum is below.

$$
\begin{aligned}
A_F = LW\left(1 + \sqrt[3]{\left(\frac{V}{V_B}\right)^2}\right) + A_B\left(1 - \sqrt[3]{\left(\frac{V}{V_B}\right)^2}\right) \\
&= LW + A_B + (LW - A_B)\sqrt[3]{\left(\frac{V}{V_B}\right)^2} \\
&= 1820 + 8316 + (1820 - 8316)\frac{\sqrt[3]{2}}{8} \\
&= 10136 - 812\sqrt[3]{2}
\end{aligned}
$$

The answer is $\mathbf{10136 - 812\sqrt[3]{2}}$.

T10. In the below number puzzle, the dark outlines define different regions, where each cell in the same region has the same nonzero digit, and no two such regions have the same digit. Each row and column define a three-digit number, with the following constraints.

Row 1: A multiple of 31

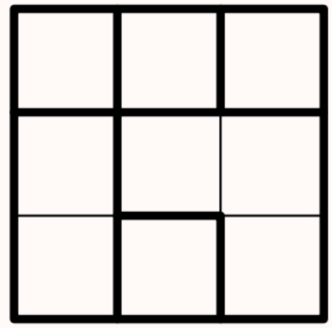
Row 2: Can be written as the sum of two 3-digit palindromes

Row 3: A perfect square

Column 1: Can be written as the sum of two squares

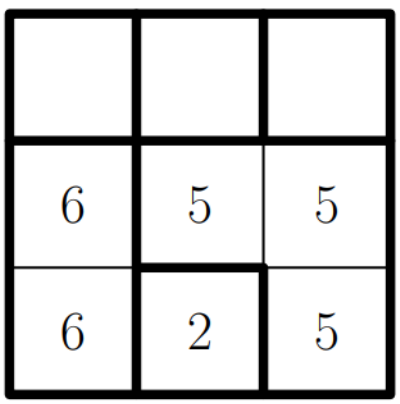
Column 2: Can be written as the product of two 2-digit integers

Column 3: Is a multiple of a 1-digit prime

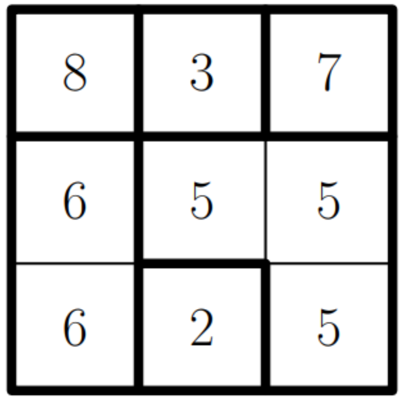


Compute the sum of the three 3-digit numbers in the three rows.

T10-Sol. $\boxed{2117}$ Focus on the second row. A three digit palindrome is of the form $\underline{a}\,\underline{b}\,\underline{a} = 101a + 10b$, so the number can be written as $101A + 10B$ for $2 \leq A \leq 9$ and $0 \leq B \leq 18$. In particular, note that $101A$ has the same units and hundreds digit, with a tens digit of 0, so adding $10B$ means the hundreds digit will either be equal or one more than the units digit. Since no two regions have the same digit, it follows that the hundreds digit will be one more than the units digit. Now focus on the third row. The only possible nonzero units digits of a perfect square are 1, 4, 5, 6, or 9, and the hundreds digit must be one more than the units digit. This immediately eliminates 9. It can be verified that no square starts with a 2 and ends with a 1, starts with a 5 and ends with a 4, or starts with a 7 and ends with a 6. This means that the last row must be 625. This information is reflected in the grid below.



(For completeness, $655 = 272 + 383$.) Note that this means column 3 is a multiple of 5 already, so that condition is satisfied. Now focus on column 1. Note that this number is equivalent to 2 modulo 4, and so in order to be the sum of two squares, both squares must be odd. However, odd squares are equivalent to 1 modulo 8, so the number in the first column must be equivalent to 2 modulo 8. This implies the top left cell is even. Previous information eliminates 2 and 6, so it must be 4 or 8. Indeed, $466 = 21^2 + 5^2$ and $866 = 29^2 + 5^2$, so this is not sufficient. Now look at row 1. The multiples of 31 starting with 4 are 403, 434, 465, and 496, all of which are impossible with the given constraints. Thus the first digit must be 8. The multiples of 31 starting with 8 are 806, 837, 868, and 899, of which only 837 is possible. Indeed, the number in the second column, 352, equals $11 \cdot 32$, so that condition is satisfied and the grid is complete. That completed grid is below.



The answer is $837 + 655 + 625 = \mathbf{2117}$.

NYSML
Power Question
60 minutes -- no calculators permitted

50 total points

2026 Championships *Heronian Triangles*

Remember that no calculators are allowed on this contest.

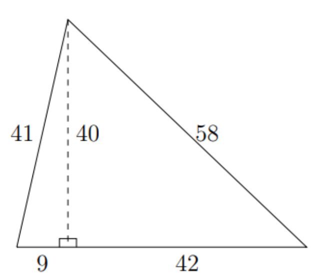
To receive full credit, the presentation must be legible, orderly, clear, and concise. When a numerical answer or formula is called for, circle or box it. Even if not completed, earlier numbered items may be used to solve later numbered items, but not vice versa. The pages submitted should be numbered in consecutive order at the top of each page.

Put your Team Number (not your Team Name) on every page you submit. ***Do not identify your team in any other way.***

BACKGROUND: This Power Question explores triangles with positive integer side lengths.
A ***primitive*** triangle is a triangle with positive integer side lengths a, b, and c such that the greatest common divisor of a, b, and c is 1. For this Power Question, a triangle with positive integer side lengths a, b, and c with $a \leq b \leq c$ is written with the shorthand (a, b, c).
A ***Pythagorean*** triangle is a right triangle with positive integer side lengths. A Pythagorean triangle (a, b, c) satisfies $c^2 = a^2 + b^2$.
A ***Heronian*** triangle is a triangle with positive integer side lengths and positive integer area. This type of triangle earns its name from Heron's formula, which relates the area of a triangle with its three side lengths. Below is Heron's formula for the area K of a triangle, where a, b, and c are the side lengths and $s = \frac{a+b+c}{2}$, is the semiperimeter. Heron's formula applies to all non-degenerate triangles, not just Heronian triangles.

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Some Heronian triangles have an altitude that splits the triangle into two Pythagorean triangles sharing a common side. This type of triangle is called a ***decomposable*** Heronian triangle. All other Heronian triangles are otherwise called ***indecomposable*** Heronian triangles. The $(41, 51, 58)$ triangle shown below with area 1020 is an example of a decomposable Heronian triangle because it is comprised of two Pythagorean triangles sharing a common side.
To ***decompose*** a Heronian triangle is to draw an altitude such that the Heronian triangle is split into two Pythagorean triangles. In the example below, the $(41, 51, 58)$ Heronian triangle is decomposed into a $(9, 40, 41)$ Pythagorean triangle with area 180 and a $(40, 42, 58)$ Pythagorean triangle with area 840. Only decomposable Heronian triangles can be decomposed.



A triangle is ***acute*** or ***obtuse*** if its largest angle is acute or obtuse. An ***isosceles*** triangle has exactly two congruent sides. A ***scalene*** triangle has three sides of different lengths.

P1. Consider the three Heronian triangles $(3, 4, 5)$, $(13, 14, 15)$, and $(20, 65, 75)$.
a) Compute the areas of each of these three triangles.
b) State, without proof, whether each triangle is decomposable and whether each triangle is primitive.
c) Decompose each of the decomposable triangles. *6 points*

P2. This problem involves finding Heronian triangles that satisfy certain conditions.
a) Find, decompose, and sketch an isosceles primitive decomposable Heronian triangle.
b) Find an obtuse primitive indecomposable Heronian triangle. Show that it cannot be decomposed. *4 points*

P3. Prove or disprove the following statements.
a) An equilateral triangle is Heronian.
b) There exists a Heronian triangle that has one $60°$ angle. *4 points*

P4. Show that the following properties are true.
a) All Pythagorean triangles are Heronian.
b) All Heronian triangles have an even perimeter. *4 points*

P5. Prove the following statements.
a) All primitive Pythagorean triangles are indecomposable.
b) All isosceles Heronian triangles are decomposable. *5 points*

P6. Not all positive integers can be the side length of a Heronian triangle, as this part of the Power Question will establish.
a) Prove that no Heronian triangle has a side length of 1.
b) Prove that no Heronian triangle has a side length of 2. *5 points*

P7. Prove or disprove whether each of the following exists.
a) A scalene primitive decomposable Heronian triangle that can be decomposed into two primitive Pythagorean triangles
b) An acute primitive indecomposable Heronian triangle *6 points*

P8. Prove that the sine and cosine of each angle of a Heronian triangle are rational numbers.
4 points

P9. Some, but not all, Heronian triangles are decomposable in more than one way.
a) Find a Heronian triangle that is decomposable in exactly two ways. Then decompose the triangle in those two ways.
b) Prove that no primitive Heronian triangle is decomposable in more than one way. *6 points*

P10. Prove that there are infinitely many Heronian triangles with side lengths that are consecutive positive integers. Then state, without proof, the smallest five Heronian triangles whose sides have lengths that are consecutive positive integers. *6 points*

NYSML

Power Question

60 minutes -- no calculators permitted

50 total points

2026 Championships *Heronian Triangles*

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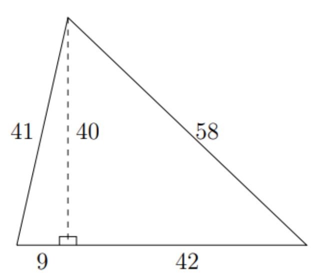
$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

Some Heronian triangles have an altitude that splits the triangle into two Pythagorean triangles sharing a common side. This type of triangle is called a ***decomposable*** Heronian triangle. All other Heronian triangles are otherwise called ***indecomposable*** Heronian triangles. The $(41, 51, 58)$ triangle shown below with area 1020 is an example of a decomposable Heronian triangle because it is comprised of two Pythagorean triangles sharing a common side.
To ***decompose*** a Heronian triangle is to draw an altitude such that the Heronian triangle is split into two Pythagorean triangles. In the example below, the $(41, 51, 58)$ Heronian triangle is decomposed into a $(9, 40, 41)$ Pythagorean triangle with area 180 and a $(40, 42, 58)$ Pythagorean triangle with area 840. Only decomposable Heronian triangles can be decomposed.



A triangle is ***acute*** or ***obtuse*** if its largest angle is acute or obtuse. An ***isosceles*** triangle has exactly two congruent sides. A ***scalene*** triangle has three sides of different lengths.

P1. Consider the three Heronian triangles $(3, 4, 5)$, $(13, 14, 15)$, and $(20, 65, 75)$.
a) Compute the areas of each of these three triangles.

b) State, without proof, whether each triangle is decomposable and whether each triangle is primitive.
c) Decompose each of the decomposable triangles. *6 points*

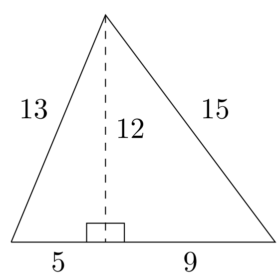
P1-Sol.

a The areas of the $(3,4,5)$ triangle, $(13,14,15)$ triangle, and $(20,65,75)$ triangle are 6, 84, and 600 respectively.

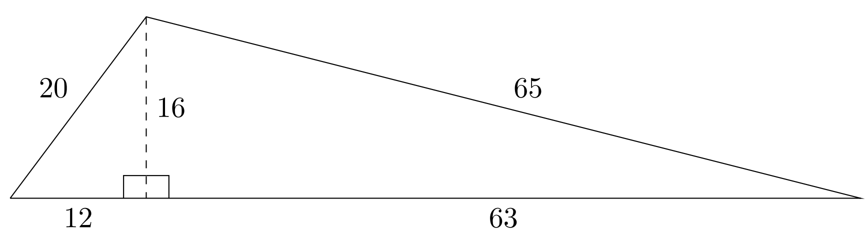
b The table below states all of the desired answers for this part.

Triangle	Decomposable	Primitive
$(3,4,5)$	No	Yes
$(13,14,15)$	Yes	Yes
$(20,65,75)$	Yes	No

c The $(13,14,15)$ triangle is decomposed into the $(9,12,15)$ Pythagorean triangle and the $(5,12,13)$ Pythagorean triangle with $9^2+12^2=15^2$ and $5^2+12^2=13^2$.



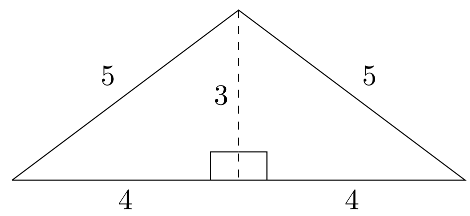
The $(20,65,75)$ triangle is decomposed into the $(12,16,20)$ Pythagorean triangle and the $(16,63,65)$ Pythagorean triangle with $12^2+16^2=20^2$ and $16^2+63^2=65^2$.



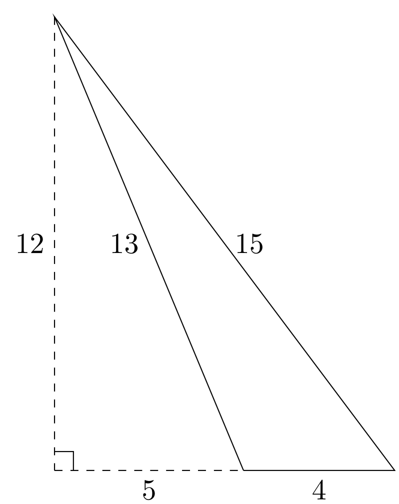
P2. This problem involves finding Heronian triangles that satisfy certain conditions.
a) Find, decompose, and sketch an isosceles primitive decomposable Heronian triangle.
b) Find an obtuse primitive indecomposable Heronian triangle. Show that it cannot be decomposed. *4 points*

P2-Sol.

a One example is the $(5,5,8)$ triangle that can be decomposed into two $(3,4,5)$ Pythagorean triangles with the shared side length being 3. The $(3,4,5)$ triangle is Pythagorean because $3^2+4^2=5^2$.



b One way to find such a triangle is to scale down the $(20,65,75)$ triangle from Problem 1 to a $(4,13,15)$ triangle. The area of this triangle is 24, which confirms it is a Heronian triangle. Furthermore, the two altitudes to the sides 13 and 15 have non-integer lengths. The third altitude to the side length 4 is outside of the triangle and therefore does not decompose it into two Pythagorean triangles.



P3. Prove or disprove the following statements.
a) An equilateral triangle is Heronian.
b) There exists a Heronian triangle that has one $60°$ angle. *4 points*

P3-Sol.

a This is false. The area of an equilateral triangle with side length t is $K=\frac{t^2\sqrt{3}}{4}$. The equation can be rewritten as $\frac{4K}{t^2}=\sqrt{3}$. The triangle cannot be Heronian otherwise the LHS would be rational while the RHS is irrational, which is impossible.

b This is false. The area of such a triangle can be calculated with the formula $K=\frac{1}{2}ab\sin 60°=\frac{ab\sqrt{3}}{2}$, where a and b are integers. The equation can be rewritten as

$\frac{2K}{ab} = \sqrt{3}$. The triangle cannot be Heronian otherwise the LHS would be rational while the RHS is irrational, which is impossible.

P4. Show that the following properties are true.
a) All Pythagorean triangles are Heronian.
b) All Heronian triangles have an even perimeter. *4 points*

P4-Sol.

a To show that a Pythagorean triangle is a Heronian triangle is equivalent to showing that all Pythagorean triangles have a positive integer area. With legs a and b, the area of a Pythagorean triangle is $K = \frac{1}{2}ab$. So if one of the legs is even, then the area is an integer. Thus, the only way for a Pythagorean triangle to have non-integer area is for both of its legs being odd, but this is impossible. Assume by way of contradiction that such a Pythagorean triangle exists with odd legs a and b and hypotenuse c. Then $a^2, b^2 \equiv 1 \pmod 4$, and by the Pythagorean Theorem, $c^2 = a^2 + b^2$, so $c^2 \equiv 2 \pmod 4$, but a perfect square cannot have a residue of 2 modulo 4. This is because all perfect squares are the square of either an even number or an odd number. The square of an even number will always be divisible by 4 and the square of an odd number will always be 1 more than multiple of 4. The assumption is invalid and all Pythagorean triangles must have at least one leg of even length and therefore have integer area and be Heronian.

b Assume by way of contradiction that there exists a Heronian triangle with an odd perimeter. Then the semiperimeter s will be rational with the numerator odd and denominator 2. Let the side lengths of such a triangle be a, b, and c, where $a + b + c$ is odd. Then by Heron's formula:

$$\begin{aligned} K &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\frac{(a+b+c)}{2}\frac{(a+b+c-2a)}{2}\frac{(a+b+c-2b)}{2}\frac{(a+b+c-2c)}{2}} \\ &= \frac{\sqrt{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}}{4} \end{aligned}$$

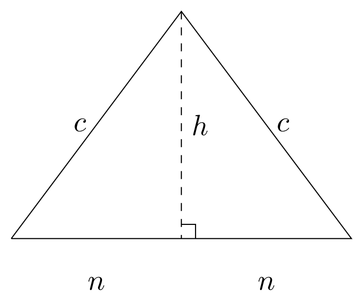
When an odd number is subtracted by an even number, the result is odd, so the numerator is the square root of the product of four odd positive integers. The product of four odd numbers is also odd, so the numerator is the square root of an odd positive integer. This means the numerator is either irrational or an odd positive integer. The triangle is Heronian, so its area is an integer, which contradicts the fact that the numerator is irrational or odd. Therefore a Heronian triangle's perimeter must be even.

P5. Prove the following statements.
a) All primitive Pythagorean triangles are indecomposable.
b) All isosceles Heronian triangles are decomposable. *5 points*

P5-Sol.

a Let a and b be the legs of a primitive Pythagorean triangle and let c be its hypotenuse. Legs a and b also serve as altitudes of the triangle. These legs, by construction, do not decompose the triangle into two right triangles. Let h_c be the altitude to hypotenuse c. This height splits the primitive Pythagorean triangle into two right triangles. Then the area $K = \frac{ch_c}{2} = \frac{ab}{2} \implies h_c = \frac{ab}{c}$. The triangle is primitive, so c does not share any factors with a or b, which makes h_c not an integer. This means the two right triangles are not Pythagorean and therefore the original primitive Pythagorean triangle is indecomposable.

b Draw isosceles Heronian triangle with legs c and base a. Draw altitude h from the apex to a. If a is odd, then the perimeter would be odd, which is impossible for a Heronian triangle as proved in Problem 4b. Then a is even, so let $a = 2n$. This splits the isosceles Heronian triangle into two congruent right triangles.



With height $h = \sqrt{c^2 - n^2}$, the area is $K = \frac{1}{2} \cdot 2n \cdot \sqrt{c^2 - n^2} = n\sqrt{c^2 - n^2}$. Rewriting the equation yields $\frac{K}{n} = \sqrt{c^2 - n^2}$. The fraction $\frac{K}{n}$ is rational, so the term $\sqrt{c^2 - n^2}$ is rational. Both c and n are integers, so $c^2 - n^2$ must be an integer. If the square root of an integer is rational, then it must be an integer. Hence h is an integer and the isosceles Heronian triangle can be decomposed into two congruent Pythagorean triangles.

P6. Not all positive integers can be the side length of a Heronian triangle, as this part of the Power Question will establish.
a) Prove that no Heronian triangle has a side length of 1.
b) Prove that no Heronian triangle has a side length of 2. *5 points*

P6-Sol.

a Assume by way of contradiction that there exists a Heronian triangle with positive integer side lengths 1, a, and b. Then by the Triangle Inequality, $|a - b| < 1$. This means $a = b$, but then the perimeter of this triangle would be odd, which is impossible for a Heronian triangle according to Problem 4b. Therefore 1 cannot be a side length of a Heronian triangle.

b Assume by way of contradiction that there exists a Heronian triangle with positive integer side lengths 2, a, and b. Then by the Triangle Inequality, $|a - b| < 2$. The non-negative difference between a and b cannot be 1 because otherwise the perimeter would be odd, which is impossible for a Heronian triangle according to Problem 4b.

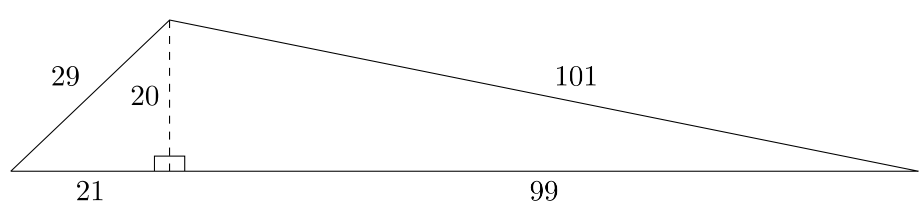
Then $a = b$, so the area of the triangle would be $K = (s - a)\sqrt{s(s-2)}$ by Heron's formula. This is a positive integer when $s(s-2)$ is a non-zero perfect square. Let that non-zero perfect square be k^2; then $s(s-2) = k^2 \implies s^2 - 2s - k^2 = 0$. This is a quadratic in terms of s. For integer solutions of s, the discriminant must be a perfect square. Let that perfect square discriminant be d^2; then $d^2 = 4 + 4k^2$. It follows that d is even so let $d = 2n$ and then $n^2 = 1 + k^2 \implies (n + k)(n - k) = 1$. Both $n + k$ and $n - k$ are distinct integers and there are no two distinct integers that multiply to 1. Therefore there are no such Heronian triangles with a side length of 2.

Alternate Solution: By the Triangle Inequality, note that $a = b$ or a and b differ by 1. If a and b differ by 1, then the perimeter of the Heronian triangle would be odd, which contradicts Problem 4b. Suppose that $a = b$. Then the Heronian triangle with side length 2 would be isosceles, and by Problem 5b, it is decomposable. If it could be decomposed by the height perpendicular to the side of length 2, then this would lead to two Pythagorean triangles with side length 1, which is a contradiction. Otherwise, the decomposition would leave a Pythagorean triangle whose hypotenuse has length 2, and this is also impossible.

P7. Prove or disprove whether each of the following exists.
a) A scalene primitive decomposable Heronian triangle that can be decomposed into two primitive Pythagorean triangles
b) An acute primitive indecomposable Heronian triangle *6 points*

P7-Sol.

a Such a triangle exists and one example is the $(29, 101, 120)$ triangle with area 1200. This triangle can be decomposed into the Pythagorean triangles $(20, 21, 29)$ and $(20, 99, 101)$ with $20^2 + 21^2 = 29^2$ and $20^2 + 99^2 = 101^2$. Drawing an accurate diagram suggests that this triangle is scalene (trigonometric calculations confirm this), and $\gcd(29, 101, 120) = 1$. Both decomposable Pythagorean triangles are primitive with $\gcd(20, 21, 29) = 1$ and $\gcd(20, 99, 101) = 1$.



b Such a triangle exists and one example is $(15, 34, 35)$. It is Heronian because it has positive integer side lengths and area 252 as found by Heron's formula. The three altitudes are $\frac{168}{5}$,

$\frac{252}{17}$, and $\frac{72}{5}$, which confirms it is indecomposable. It is acute because $15^2 + 34^2 > 35^2$, and it is primitive because $\gcd(15, 34, 35) = 1$.

P8. Prove that the sine and cosine of each angle of a Heronian triangle are rational numbers.

4 points

P8-Sol. Define Heronian triangle ABC with $a = BC, b = CA, c = AB$. Then, let K be the area of $\triangle ABC$. Using the area formula with sine gives three equations.

$$K = \frac{1}{2}ab\sin C \implies \frac{2K}{ab} = \sin C$$
$$K = \frac{1}{2}bc\sin A \implies \frac{2K}{bc} = \sin A$$
$$K = \frac{1}{2}ac\sin B \implies \frac{2K}{ac} = \sin B$$

Because the area of the triangle is an integer and the side lengths are positive integers, it follows that the sines of the three angles are rational.
Using the Law of Cosines, a similar argument can be made.

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

The right-hand sides of all three equations are rational, so the cosines of the three angles are also all rational.

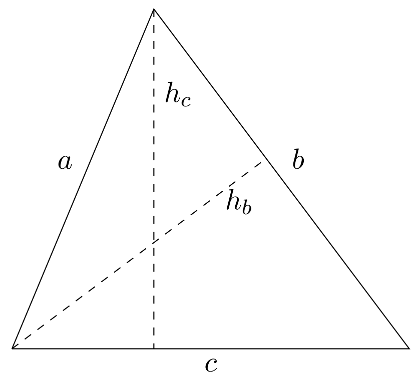
P9. Some, but not all, Heronian triangles are decomposable in more than one way.
a) Find a Heronian triangle that is decomposable in exactly two ways. Then decompose the triangle in those two ways.
b) Prove that no primitive Heronian triangle is decomposable in more than one way. *6 points*

P9-Sol.

a There are many examples. One example is the $(65, 70, 75)$ triangle, which is similar to the $(13, 14, 15)$ triangle given in Problem 1. The $(65, 70, 75)$ triangle can be decomposed one way into the Pythagorean triangles $(25, 60, 65)$ and $(45, 60, 75)$ with $25^2 + 60^2 = 65^2$ and $45^2 + 60^2 = 75^2$. It can also be decomposed into the Pythagorean triangles $(42, 56, 70)$ and

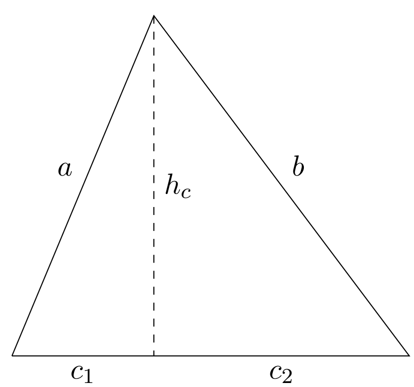
$(33, 56, 65)$ with $42^2 + 56^2 = 70^2$ and $33^2 + 56^2 = 65^2$. Note that this triangle cannot be decomposed in a third way because its area is 2100 and the altitude to the side with length 65 is $\frac{840}{13}$, which is not an integer.

b First, start with a Heronian triangle (a, b, c) that is decomposable in more than one way. Such a triangle is guaranteed to exist from the problem statement and from Problem 8a. Then this triangle must have at least two positive integer heights. Without loss of generality, let the heights h_b and h_c be the positive integer heights that are perpendicular to sides b and c respectively. A diagram is shown below.



The area of the triangle is $K = \frac{bh_b}{2} = \frac{ch_c}{2} \implies bh_b = ch_c$. It is impossible for b and c to be relatively prime. Assume for the sake of contradiction that b and c are relatively prime. Then b divides h_c and c divides h_b. This means $h_b = ch$ and $h_c = bh$, where h is a positive integer. Then $h_c = bh \geq b$, which is impossible because h_c and b are sides of a right triangle with h_c a leg and b a hypotenuse. A leg is always shorter than the hypotenuse.

Then $\gcd(b, c) = g > 1$ and let $b = b'g$ and $c = c'g$ for positive integers b' and c' with $\gcd(b', c') = 1$. Using the same area argument, $b'gh_b = c'gh_c \implies b'h_b = c'h_c$. Then $h_b = c'h$ and $h_c = b'h$, where $h = \frac{h_c}{b'} = \frac{h_b}{c'}$ is a positive integer. Height h_c splits side c into two segments c_1 and c_2 as shown in the diagram below.



Using the Pythagorean Theorem twice yields the following two equations:

$$c_1^2 = a^2 - h_c^2$$

$$c_2^2 = b^2 - h_c^2$$

Both b and h_c are divisible by b', so c_2 is also divisible by b'. Set $c_2 = b'k$ for some integer k. Then the second equation can be simplified by:

$$b^2 = h_c^2 + c_2^2 \implies (b'g)^2 = (b'h)^2 + (b'k)^2 \implies g^2 = h^2 + k^2$$

Applying the Pythagorean Theorem one more time yields:

$$\begin{aligned} a^2 &= h_c^2 + c_1^2 \\ &= h_c^2 + (c - c_2)^2 \\ &= h_c^2 + (c - b'k)^2 \\ &= b'^2(h^2 + k^2) + c^2 - 2cb'k \\ &= b'^2 g^2 + c_1^2 g^2 - 2c_1 g b'k \\ &= g(b'^2 g + c_1^2 g - 2b'c_1 k) \end{aligned}$$

Therefore a^2 is divisible by g, which means a is divisible by g. Then a, b, and c are all divisible by g and the Heronian triangle (a, b, c) that is decomposable in multiple ways is not primitive.

P10. Prove that there are infinitely many Heronian triangles with side lengths that are consecutive positive integers. Then state, without proof, the smallest five Heronian triangles whose sides have lengths that are consecutive positive integers. *6 points*

P10-Sol. Given a triangle with side lengths $n-1, n$, and $n+1$, where n is a positive integer, the semi-perimeter s is $s = \frac{3n}{2}$. The area of the triangle is below.

$$\begin{aligned} K &= \sqrt{\frac{3n}{2}\left(\frac{3n}{2} - (n-1)\right)\left(\frac{3n}{2} - n\right)\left(\frac{3n}{2} - (n+1)\right)} \\ &= \sqrt{\frac{3n}{2}\left(\frac{n+2}{2}\right)\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)} \\ &= \frac{n}{4}\sqrt{3(n+2)(n-2)} \end{aligned}$$

For the area to be an integer, n must be even, so let $n = 2m$. Then the area is $K = m\sqrt{3(m+1)(m-1)}$. The inside of the square root must be a perfect square, so let $3(m+1)(m-1) = t^2$, for an integer t. Note that t is a multiple of 3, so let $t = 3k$ and then $(m+1)(m-1) = 3k^2$ which simplifies to:

$$m^2 - 3k^2 = 1.$$

The equation $m^2 - 3k^2 = 1$ is known as a Pell equation. A Pell equation has infinitely many solutions, which means there are an infinite number of Heronian triangles with consecutive sides.

One does not need to have knowledge of Pell equations to solve this, however. The rest of this solution is dedicated to solving this without a Pell equation.
By inspection, the minimal solution is $(m_1, k_1) = (2, 1)$. Then notice that

$$\left(m_1^2 - 3k_1^2\right)^2 = 1^2 \implies (m_1^2 + 3k_1^2)^2 - 3(2m_1k_1)^2 = 1.$$

This creates another solution $(m_2, k_2) = (m_1^2 + 3k_1^2, 2m_1k_1) = (7, 4)$. To create the third solution, compute the following:

$$(m_2^2 - 3k_2^2)(m_1^2 - 3k_1^2) = 1$$
$$(m_1^2m_2^2 - 3m_1^2k_2^2 - 3m_1^2k_2^2 + 9k_1^2k_2^2) = 1$$
$$(m_1m_2 + 3k_1k_2)^2 - 3(m_1k_2 + m_2k_1)^2 = 1$$

The third solution is $(m_3, k_3) = (m_1m_2 + 3k_1k_2, m_1k_2 + m_2k_1) = (26, 15)$. In general, to find the solution (m_{i+1}, k_{i+1}) given (m_1, k_1) and (m_i, k_i), compute the following:

$$(m_i^2 - 3k_i^2)(m_1^2 - 3k_1^2) = 1$$
$$(m_i^2m_1^2 - 3m_i^2k_1^2 - 3m_1^2k_i^2 + 9k_i^2k_1^2) = 1$$
$$(m_im_1 + 3k_ik_1)^2 - 3(m_1k_i + m_ik_1)^2 = 1$$

Thus $(m_{i+1}, k_{i+1}) = (m_im_1 + 3k_ik_1, m_1k_i + m_ik_1)$ given $(m_1, k_1) = (2, 1)$ creates a recursive definition detailing an infinite number of solutions to the equation $m^2 - 3k^2 = 1$, therefore an infinite number of Heronian triangles with consecutive side length.
This recurrence relationship suggests an explicit exponential relationship. This can be found by raising the equation $m_1^2 - 3k_1^2 = (m_1 - k_1\sqrt{3})(m_1 + k_1\sqrt{3}) = 1$ to the ith integer power which yields:

$$(m_1^2 - 3k_1^2)^i = (m_1 - k_1\sqrt{3})^i(m_1 + k_1\sqrt{3})^i = m_i^2 - 3k_i^2 = (m_i - k_i\sqrt{3})(m_i + k_i\sqrt{3})$$
$$m_i - k_i\sqrt{3} = (m_1 - k_1\sqrt{3})^i$$
$$m_i + k_i\sqrt{3} = (m_1 + k_1\sqrt{3})^i$$

Knowing that the minimal solution is $(m_1, k_1) = (2, 1)$, this system of equations in terms of m_i and k_i can be solved with the following solutions.

$$m_i = \frac{(m_1 + k_1\sqrt{3})^i + (m_1 - k_1\sqrt{3})^i}{2} = \frac{(2 + \sqrt{3})^i + (2 - \sqrt{3})^i}{2}$$
$$k_i = \frac{(m_1 + k_1\sqrt{3})^i - (m_1 - k_1\sqrt{3})^i}{2\sqrt{3}} = \frac{(2 + \sqrt{3})^i - (2 - \sqrt{3})^i}{2\sqrt{3}}$$

Using either the recursive formula or the explicit formula along with Problem 1, the 5 smallest triangles are as follows:

i	Triangle
1	$(3, 4, 5)$
2	$(13, 14, 15)$
3	$(51, 52, 53)$
4	$(193, 194, 195)$
5	$(723, 724, 725)$

NYSML

Individual Round

10 min/pair -- no calculators permitted

1 point each -- 150 total points

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

I1. In rectangle $RECT$, $RE = 20 < EC$. Point D is on $\overline{EC}$ such that $RD = RT$. Given that $DC = 4$, compute RT.

I2. Suppose that a and b are nonzero real numbers and $a^2 + b^2 = 6ab$. Compute the greatest possible value of $\frac{a+b}{a-b}$.

I3. The nonnegative decimal digits A and B satisfy

$$(\underline{2}\,\underline{8}\,\underline{A}\,\underline{6})^2 = \underline{8}\,\underline{1}\,\underline{5}\,\underline{B}\,\underline{7}\,\underline{3}\,\underline{6}.$$

Compute $A + B$.

I4. Alice and Bob play a game, taking turns flipping a fair coin, with Alice going first. The first player to flip heads immediately after the other player flipped heads wins. Compute the probability Alice wins.

I5. Given that M and N are positive integers satisfying $M^{20}N^{26} = 2^{2026}$, compute the least possible integer value of $\frac{M}{N}$.

I6. Suppose that $\triangle ABC$ has a right angle at C, and suppose that $AC = 20$ and $AB = 25$. Vertex C is on a circle with center O, and the circle is tangent to $\overline{AB}$ at G. Given that chord $\overline{GC}$ has length 13, compute the area of $\triangle GOC$.

I7. Positive integers a, b, c, and d satisfy $\log_a b = \frac{5}{2}$ and $\log_c d = \frac{5}{4}$. Given that $a - c = 9$, compute $b - d$.

I8. The four-digit base-ten number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is a multiple of 13, where A, B, C, and D are not necessarily distinct digits. The sum of its digits is also a multiple of 13. The base-eight number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}_{\text{eight}}$ is also a multiple of thirteen. Compute **both** possible values of $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$.

I9. The graphs of $y = 3x$, $y = kx + 5$, and $y = -\frac{1}{3}x$ enclose a triangle of area 25. Given that $-\frac{1}{3} < k < 3$, compute the least possible value of k.

I10. Compute the value of x that minimizes

$$|x - 100| + |2x - 99| + |3x - 98| + \cdots + |98x - 3| + |99x - 2| + |100x - 1|.$$

2026 Championships

I1. In rectangle $RECT$, $RE = 20 < EC$. Point D is on $\overline{EC}$ such that $RD = RT$. Given that $DC = 4$, compute RT.

I2. Suppose that a and b are nonzero real numbers and $a^2 + b^2 = 6ab$. Compute the greatest possible value of $\frac{a+b}{a-b}$.

2026 Championships

I1) ______ **I2)** ______

Student Name:____________________ **Team Name:**______________

2026 Championships

I3. The nonnegative decimal digits A and B satisfy

$$(\underline{2}\,\underline{8}\,\underline{A}\,\underline{6})^2 = \underline{8}\,\underline{1}\,\underline{5}\,\underline{B}\,\underline{7}\,\underline{3}\,\underline{6}.$$

Compute $A + B$.

I4. Alice and Bob play a game, taking turns flipping a fair coin, with Alice going first. The first player to flip heads immediately after the other player flipped heads wins. Compute the probability Alice wins.

2026 Championships

I3) [ ]

I4) [ ]

Student Name:____________________ **Team Name:**_______________

2026 Championships

I5. Given that M and N are positive integers satisfying $M^{20}N^{26} = 2^{2026}$, compute the least possible integer value of $\frac{M}{N}$.

I6. Suppose that $\triangle ABC$ has a right angle at C, and suppose that $AC = 20$ and $AB = 25$. Vertex C is on a circle with center O, and the circle is tangent to $\overline{AB}$ at G. Given that chord $\overline{GC}$ has length 13, compute the area of $\triangle GOC$.

2026 Championships

I5) [ ]

I6) [ ]

Student Name:____________________ Team Name:_______________

2026 Championships

I7. Positive integers a, b, c, and d satisfy $\log_a b = \frac{5}{2}$ and $\log_c d = \frac{5}{4}$. Given that $a - c = 9$, compute $b - d$.

I8. The four-digit base-ten number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is a multiple of 13, where A, B, C, and D are not necessarily distinct digits. The sum of its digits is also a multiple of 13. The base-eight number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}_{\text{eight}}$ is also a multiple of thirteen. Compute **both** possible values of $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$.

2026 Championships

I7) [ ]

I8) [ ]

Student Name:____________________ **Team Name:**_______________

2026 Championships

I9. The graphs of $y = 3x$, $y = kx + 5$, and $y = -\frac{1}{3}x$ enclose a triangle of area 25. Given that $-\frac{1}{3} < k < 3$, compute the least possible value of k.

I10. Compute the value of x that minimizes

$$|x - 100| + |2x - 99| + |3x - 98| + \cdots + |98x - 3| + |99x - 2| + |100x - 1|.$$

NYSML Individual Round

10 min/pair -- no calculator permitted

1 point each -- 150 total points

The word "compute" calls for an exact answer in simplest form.

2026 Championships

I9)

I10)

Student Name:______________________ **Team Name:**_________________

NYSML

Individual Round

10 min/pair -- no calculators permitted

1 point each -- 150 total points

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

I1. In rectangle $RECT$, $RE = 20 < EC$. Point D is on $\overline{EC}$ such that $RD = RT$. Given that $DC = 4$, compute RT.

I1-Sol. $\boxed{52}$ Let $RD = RT = x$. Then it follows that $ED = x - 4$. By the Pythagorean Theorem applied to $\triangle RED$, it follows that $(x-4)^2 + 20^2 = x^2$, which implies $x^2 - 8x + 16 + 400 = x^2$, and so $8x = 416 \rightarrow x = RT = \mathbf{52}$.

I2. Suppose that a and b are nonzero real numbers and $a^2 + b^2 = 6ab$. Compute the greatest possible value of $\frac{a+b}{a-b}$.

I2-Sol. $\boxed{\sqrt{2}}$ Square $\frac{a+b}{a-b}$ to obtain $\left(\frac{a+b}{a-b}\right)^2 = \frac{a^2+b^2+2ab}{a^2+b^2-2ab}$, which is equivalent to $\frac{6ab+2ab}{6ab-2ab} = \frac{8ab}{4ab} = 2$. Thus the two possible values of $\frac{a+b}{a-b}$ are $\pm\sqrt{2}$, and the greater of these is $\boldsymbol{\sqrt{2}}$.

I3. The nonnegative decimal digits A and B satisfy

$$(\underline{2}\,\underline{8}\,\underline{A}\,\underline{6})^2 = \underline{8}\,\underline{1}\,\underline{5}\,\underline{B}\,\underline{7}\,\underline{3}\,\underline{6}.$$

Compute $A + B$.

I3-Sol. $\boxed{11}$ The last two digits of $(\underline{A}\,\underline{6})^2$ must be $\underline{3}\,\underline{6}$, i.e., the last two digits of $100A^2 + 120A + 36$ must be $\underline{3}\,\underline{6}$. Therefore A is 0 or 5. If $A = 0$, then $\underline{2}\,\underline{8}\,\underline{A}\,\underline{6}$ is so close to 2800 that its square could not possibly start with the digit 8, so A must be 5.
The sum of the digits of $\underline{2}\,\underline{8}\,\underline{5}\,\underline{6}$ is 21, so its square is a multiple of 9. Therefore the sum of the digits of the right-hand side of the original equation is also a multiple of 9. That sum of digits is

$$8 + 1 + 5 + B + 7 + 3 + 6 = 30 + B.$$

In order for this to be a multiple of 9, the value of B must be 6. Therefore $A + B = 5 + 6 = \mathbf{11}$.

I4. Alice and Bob play a game, taking turns flipping a fair coin, with Alice going first. The first player to flip heads immediately after the other player flipped heads wins. Compute the probability Alice wins.

I4-Sol. $\boxed{\frac{2}{5} \text{ or } 0.4 \text{ or } 40\%}$ Let Alice win with probability p. If she flips a tails on her first flip, then the game resets with Alice now going second, so she has $1-p$ probability of winning. If she flips heads and Bob flips tails, then the game resets, while if she flips heads and Bob flips heads, Alice loses. Thus it follows that

$$p = \frac{1}{2}(1-p) + \frac{1}{4}p = \frac{1}{2} - \frac{1}{4}p,$$

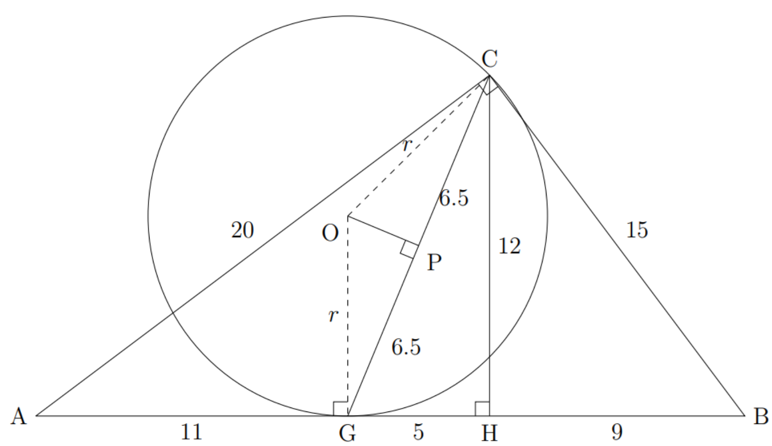
which has the solution $p = \mathbf{\frac{2}{5}}$.

I5. Given that M and N are positive integers satisfying $M^{20}N^{26} = 2^{2026}$, compute the least possible integer value of $\frac{M}{N}$.

I5-Sol. $\boxed{128}$ Note that M and N must be powers of 2, so let $M = 2^m$ and $N = 2^n$, so $20m + 26n = 2026$, i.e., $10m + 13n = 1013$. In order for $\frac{M}{N}$ to be an integer, it must be that $m \geq n$. By considering residues modulo 10, it follows that n must be equivalent to 1 mod 10 and it can be verified that $(100, 1)$ is a solution. If n is increased by 10, then another solution can be found by decreasing m by 13. Thus other solutions are $(87, 11)$, $(74, 21)$, $(61, 31)$, $(48, 41)$, $(35, 51)$, $(22, 61)$, and $(9, 71)$. The pair which has m and n closest together is $(48, 41)$, making the answer $\frac{2^{48}}{2^{41}} = 2^7 = \mathbf{128}$.

I6. Suppose that $\triangle ABC$ has a right angle at C, and suppose that $AC = 20$ and $AB = 25$. Vertex C is on a circle with center O, and the circle is tangent to $\overline{AB}$ at G. Given that chord $\overline{GC}$ has length 13, compute the area of $\triangle GOC$.

I6-Sol. $\boxed{\frac{845}{48} \text{ or } 17\frac{29}{48} \text{ or } 17.6041\overline{6}}$ Consider the diagram below.



Let H be the foot of the altitude from C onto side $\overline{AB}$. By the Pythagorean Theorem, $BC = 15$. Then $CH = \frac{15\cdot 20}{25} = 12$ from calculating the area of $\triangle ABC$ in two ways.

Let P be the foot of the altitude from O onto side $\overline{GC}$. Then $GP = PC$ and $\triangle OPG \cong \triangle OPC$. Both $\overline{OG}$ and $\overline{CH}$ are perpendicular to $\overline{AB}$, so $\overline{OG} \parallel \overline{CH}$. With transversal $\overline{GC}$, it follows that $\angle OGC \cong \angle GCH$ which makes $\triangle OPG \sim \triangle GHC$. The area of $\triangle GOC$ can be calculated as follows.

$$\begin{aligned}[GOC] &= [OPG] + [OPC] \\ &= 2[OPG] \\ &= 2[GCH]\left(\frac{6.5}{12}\right)^2 \\ &= 2\left(\frac{12\cdot 5}{2}\right)\left(\frac{13}{24}\right)^2 \\ &= \frac{845}{48}.\end{aligned}$$

Thus the answer is $[GOC] = \mathbf{\frac{845}{48}}$.

I7. Positive integers a, b, c, and d satisfy $\log_a b = \frac{5}{2}$ and $\log_c d = \frac{5}{4}$. Given that $a - c = 9$, compute $b - d$.

I7-Sol. $\boxed{3093}$ The first given equation implies $b = a^{5/2}$ and so $b^2 = a^5$. The second given equation similarly implies $d^4 = c^5$. Because a, b, c, and d are all integers, it follows that $a = r^2$ and $c = s^4$ for some positive integers r and s. Subtract to obtain $a - c = r^2 - s^4 = 9$, which implies $(r - s^2)(r + s^2) = 9$. Thus it follows that $r - s^2 = 1$ and $r + s^2 = 9$, which implies $r = 5$ and $s = 2$. The answer is $b - d = r^5 - s^5 = 5^5 - 2^5 = \mathbf{3093}$.

I8. The four-digit base-ten number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ is a multiple of 13, where A, B, C, and D are not necessarily distinct digits. The sum of its digits is also a multiple of 13. The base-eight number $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}_{\text{eight}}$ is also a multiple of thirteen. Compute **both** possible values of $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$.

I8-Sol. $\boxed{4225 \text{ and } 6331\ (\textit{need both})}$ The problem statement implies $13|(1000A + 100B + 10C + D)$, which implies $13 \mid (12A + 9B + 10C + D)$ by reducing modulo 13. It is also given that $13 \mid (A + B + C + D)$, and so 13 divides the difference $11A + 8B + 9C$.
Note also that $13 \mid (512A + 64B + 8C + D)$, which implies $13 \mid (5A + 12B + 8C + D)$. Again using the difference, it follows that $13 \mid (4A + 11B + 7C)$.
Thus it follows that $13 \mid (7(11A + 8B + 9C) - 9(4A + 11B + 7C))$, which is equivalent to $13 \mid (41A - 43B)$, which implies $13 \mid (2A - 4B)$. Because 2 is a factor of $2A - 4B$ but not of 13, it follows that $13 \mid (A - 2B)$.

Because A and B are base-eight digits, the possibilities for (A, B) are $(2, 1)$, $(4, 2)$, and $(6, 3)$. If $(A, B) = (2, 1)$, then it follows that $C = 1$ and $D = 9$, but $D = 9$ is not a base-eight digit, and so this is a contradiction. If $(A, B) = (4, 2)$, then it follows that $C = 2$ and $D = 5$. If $(A, B) = (6, 3)$, then it follows that $C = 3$ and $D = 1$. Thus the two possible values of $\underline{A}\,\underline{B}\,\underline{C}\,\underline{D}$ are **4225** and **6331**.

I9. The graphs of $y = 3x$, $y = kx + 5$, and $y = -\frac{1}{3}x$ enclose a triangle of area 25. Given that $-\frac{1}{3} < k < 3$, compute the least possible value of k.

I9-Sol. $\boxed{\frac{4-\sqrt{10}}{3}}$ The graph of $y = kx + 5$ intersects the other two lines at $\left(\frac{5}{3-k}, \frac{15}{3-k}\right)$ and $\left(-\frac{15}{3k+1}, \frac{5}{3k+1}\right)$. Their distances to the origin are $\frac{5\sqrt{10}}{3-k}$ and $\frac{5\sqrt{10}}{3k+1}$, respectively. Because the triangle formed is a right triangle, it follows that $\left(\frac{5\sqrt{10}}{3-k}\right)\left(\frac{5\sqrt{10}}{3k+1}\right) = 50$. The solutions to the resulting quadratic are $k = \frac{4\pm\sqrt{10}}{3}$. Both of these solutions are in the desired range, and so the least possible k is $\mathbf{\frac{4-\sqrt{10}}{3}}$.

I10. Compute the value of x that minimizes

$$|x - 100| + |2x - 99| + |3x - 98| + \cdots + |98x - 3| + |99x - 2| + |100x - 1|.$$

I10-Sol. $\boxed{\frac{30}{71}}$ This solution solves the problem situation in a more general form. The first claim is that

$$\sum_{i=1}^{n} |x - a_i|$$

is minimized when x is equal to the median of $\{a_1, a_2, \ldots, a_n\}$. Without loss of generality, suppose $a_1 \le a_2 \le \cdots \le a_n$, and imagine moving x across the number line. When $x < a_1$, think about moving on the graph along a line of slope $-n$. When $a_1 < x < a_2$, think about moving along a line of slope $-n + 2$. The slope increases on every interval, but it switches from negative to positive at the median, meaning that is where the minimum occurs. Write $|kx - (101 - k)|$ as $k \cdot \left|x - \frac{101-k}{k}\right|$. Thus, the answer is the median of the list of numbers

$$100, \frac{99}{2}, \frac{99}{2}, \frac{98}{3}, \frac{98}{3}, \frac{98}{3}, \ldots, \underbrace{\frac{1}{100}, \ldots, \frac{1}{100}}_{100 \text{ times}}.$$

There are $\frac{100\cdot101}{2} = 5050$ numbers in this list. There are $\frac{k(k+1)}{2}$ fractions with denominator at most k, so it suffices to find the least k satisfying

$$\frac{k(k+1)}{2} \ge \frac{5050}{2} \implies k(k+1) \ge 5050.$$

An estimate of 70 (as $70^2 = 4900$) is a bit under, but $k = 71$ will do. Thus the answer is $x = \mathbf{\frac{30}{71}}$.

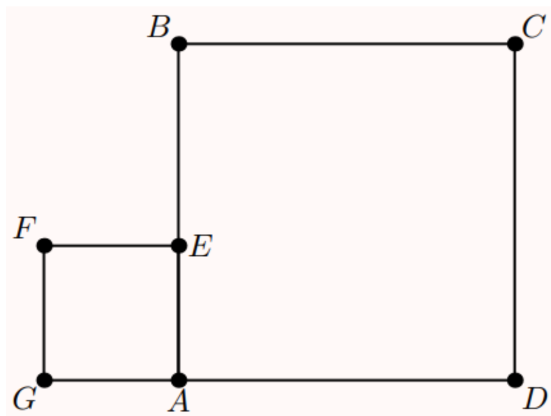
NYSML
Relay Round
no calculators

Each Round: sub-teams of 3 -- 5 points at 3 minutes, 3 points at 6 minutes

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. Suppose that $N-2$, $N-1$, and N are all positive integers that can be written as the product of two distinct primes. Compute the least possible value of N.

R1-2. *Let N be the number you will receive.* Squares $ABCD$ and $AEFG$ are on the plane, with E on $\overline{AB}$, as shown.



Given that $CF = 50$ and $AB = N$, compute AE.

R1-3. *Let N be the number you will receive.* Stan writes down the integer M on a piece of paper. Then he repeatedly writes down the product of the digits of the previous number until he writes down a single-digit number. For example, starting with 68, he would then write down 48, then 32, and finally 6. Compute the least possible $M > 2026$ such that the last number Stan writes down is N.

R2-1. Compute the units digit of $(2^3 + 0^3 + 2^3 + 6^3)^3$.

R2-2. *Let N be the number you will receive.* The sequence a_n satisfies

$$a_{n+1} = a_n + 2n + 9$$

for all positive integers n, and $a_0 = 16$. Compute a_N.

R2-3. *Let N be the number you will receive.* A textbook is made by layering 250 sheets of paper, folding them in half to create the book, and labeling the front and back of each half-sheet with a page number so the pages count up from 1 to 1000. For example, the bottom sheet of paper is labeled with the numbers 1, 2, 999, and 1000. The sheet of paper with the page labeled N has four page labels on it. Compute the positive difference between the greatest and least of those four labels.

R1-1. Suppose that $N-2$, $N-1$, and N are all positive integers that can be written as the product of two distinct primes. Compute the least possible value of N.

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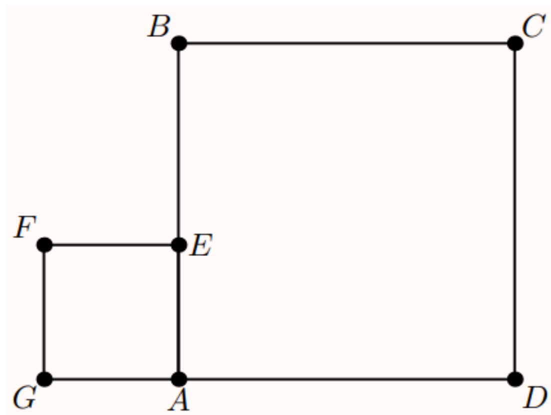

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NYSML

Relay Round

no calculators

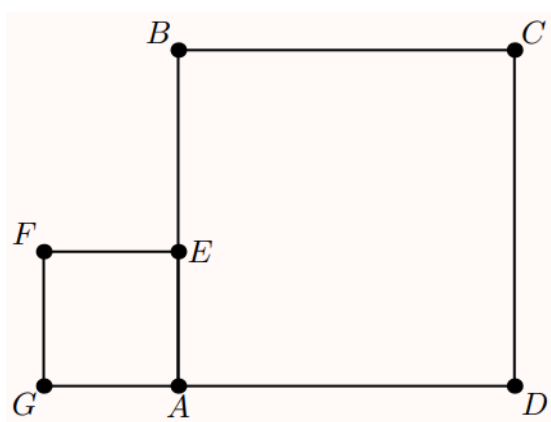
Each Round: sub-teams of 3 -- 5 points at 3 minutes, 3 points at 6 minutes

2026 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. Suppose that $N-2$, $N-1$, and N are all positive integers that can be written as the product of two distinct primes. Compute the least possible value of N.

R1-1-Sol. $\boxed{35}$ One can write down positive integers until one finds the answer, but here is a more systematic method. Note that only one of the three numbers can be even, because otherwise one of the numbers would be a multiple of 4. Thus the least and greatest of the three numbers must be odd, and the middle one is even. Note that at most one of the numbers is a multiple of 3, and so one of the numbers is the product of two primes which are not 2 or 3. The least such number is $5 \cdot 7 = 35$, and sure enough, 33, 34, and 35 satisfy the condition, making the answer **35**.

R1-2. *Let N be the number you will receive.* Squares $ABCD$ and $AEFG$ are on the plane, with E on $\overline{AB}$, as shown.



Given that $CF = 50$ and $AB = N$, compute AE.

R1-2-Sol. $\boxed{5}$ Let $AE = K$. Here are two ways to solve this problem. One can compute $DG = N + K$ and $BE = N - K$, so it follows that

$$CF^2 = (N + K)^2 + (N - K)^2 = 2(N^2 + K^2).$$

Alternatively, notice that AFC is a right triangle with right angle at A, with $AF = K\sqrt{2}$ and $AC = N\sqrt{2}$, and thus

$$CF^2 = 2N^2 + 2K^2.$$

Either way, it follows that

$$K = \sqrt{\frac{CF^2 - 2N^2}{2}} = \sqrt{\frac{50^2 - 2N^2}{2}} = \sqrt{1250 - N^2}.$$

With $N = 35$, $N^2 = 1225$ and $K = \mathbf{5}$.

R1-3. *Let N be the number you will receive.* Stan writes down the integer M on a piece of paper. Then he repeatedly writes down the product of the digits of the previous number until he writes down a single-digit number. For example, starting with 68, he would then write down 48, then 32, and finally 6. Compute the least possible $M > 2026$ such that the last number Stan writes down is N.

R1-3-Sol. $\boxed{3115}$ Note that having a 0 in the number instantly makes the product 0, so if $N = 0$, then $M = 2027$ would work.
Otherwise, the digit 0 cannot appear, so it follows that $M \geq 2111$. In fact, for $N = 2, 4, 6, 8$, the answers are $M = 2111, 2112, 2113, 2114$, respectively.
The problem is more interesting when N is odd. Note that if the number has an even digit, then the product of its digits will be even, implying that the last digit is even. Thus, in order to end with an odd digit, there must be no even digit in the number. This means $M \geq 3111$. This implies that the answers for $N = 3, 5, 9$ are 3111, 3115, and 3113, respectively. If $N = 1$ or $N = 7$, the problem becomes substantially harder to track; luckily $N = 5$, and so the answer is **3115**.
(For the curious, for $T = 1$ and $T = 7$, the answers are 11111 and 7111, respectively. Try proving this!)

R2-1. Compute the units digit of $(2^3 + 0^3 + 2^3 + 6^3)^3$.

R2-1-Sol. $\boxed{8}$ For any positive integer n, the units digit of n^3 is entirely determined by the units digit of n, so it suffices to compute only the units digits of intermediate computations. The cubes have units digits

$$\begin{aligned} 2^3 &= 8 \to 8, \\ 0^3 &= 0 \to 0, \\ 6^3 &= 216 \to 6. \end{aligned}$$

Therefore the units digit of $2^3 + 0^3 + 2^3 + 6^3$ is that of $8 + 8 + 0 + 6 = 22$, so the units digit is 2. The units digit of $(2^3 + 0^3 + 2^3 + 6^3)^3$ is that of 2^3, which is **8**.

R2-2. *Let N be the number you will receive.* The sequence a_n satisfies

$$a_{n+1} = a_n + 2n + 9$$

for all positive integers n, and $a_0 = 16$. Compute a_N.

R2-2-Sol. $\boxed{144}$ Evaluate the first few terms of the sequence to find a pattern:

$$\begin{aligned} a_1 &= 16 + 0 + 9 = 25, \\ a_2 &= 25 + 2 + 9 = 36, \\ a_3 &= 36 + 4 + 9 = 49, \\ a_4 &= 49 + 6 + 9 = 64. \end{aligned}$$

Therefore $a_n = (n+4)^2$ for the values $0 \le n \le 4$. If the pattern holds, then that would imply that $a_8 = (8+4)^2 = 144$. The pattern can be seen in multiple ways. First, observationally, notice that as the value of n increases from 0, the expression $2n + 9$ attains every odd value from 9 in sequence, and because the sum of the first m odd numbers is always m^2, the pattern must hold in general.

Formally, one can show $a_n = (n+4)^2$ using induction. The base case $n = 0$ is given, and the base cases $1 \le n \le 4$ are shown above. For the inductive step, assume that $a_m = (m+4)^2$ for some $m \ge 4$. Then

$$\begin{aligned} a_{m+1} &= a_m + 2m + 9 \\ &= (m+4)^2 + 2m + 9 \\ &= m^2 + 8m + 16 + 2m + 9 \\ &= m^2 + 10m + 25 \\ &= (m+5)^2. \end{aligned}$$

This proves the inductive step which completes the proof. In summary $a_8 = \mathbf{144}$.

R2-3. *Let N be the number you will receive.* A textbook is made by layering 250 sheets of paper, folding them in half to create the book, and labeling the front and back of each half-sheet with a page number so the pages count up from 1 to 1000. For example, the bottom sheet of paper is labeled with the numbers 1, 2, 999, and 1000. The sheet of paper with the page labeled N has four page labels on it. Compute the positive difference between the greatest and least of those four labels.

R2-3-Sol. $\boxed{715}$ The sheet one up from the bottom will be labeled with the page numbers 3, 4, 997, and 998. In general, the sheet containing the even page number n will also have the numbers $n-1$, $1001-n$, and $1002-n$, and the difference between the greatest and least numbers is $(1002-n)-(n-1)=1003-2n$. The sheet containing the odd page number m will also have the numbers $m+1$, $1000-m$, and $1001-m$, and the difference between the greatest and least numbers is $(1001-m)-m=1001-2m$.
With $N=144$, the requested difference is $1003-2\cdot 144=715$. The page numbers on the sheet are 143, 144, 857, and 858, with $858-143=$ **715**.

TIEBREAKER 1

For each positive integer n, let $f(n)$ denote the sum of the first n positive perfect squares. For example, $f(3) = 1 + 4 + 9 = 14$. Compute the least positive integer m for which $f(m)$ and $f(2026)$ have the same remainder upon division by 2026.

Student Name:______________________

Team Name:______________________

TB-1 Answer:

TIEBREAKER 1

For each positive integer n, let $f(n)$ denote the sum of the first n positive perfect squares. For example, $f(3) = 1 + 4 + 9 = 14$. Compute the least positive integer m for which $f(m)$ and $f(2026)$ have the same remainder upon division by 2026.

$\boxed{506}$ Recall the formula for the sum of the first n positive perfect squares:

$$f(n) = \frac{n(n+1)(2n+1)}{6}.$$

Therefore

$$f(2026) = \frac{2026 \cdot 2027 \cdot 4053}{6} = 1013 \cdot 2027 \cdot 1351.$$

Because $1013 \mid f(2026)$, it follows that $f(2026)$ is equivalent to either 0 or 1013 modulo 2026, and because $f(2026)$ is odd, it must be that $f(2026) \equiv 1013 \bmod 2026$. Therefore the answer is the least positive integer m for which $f(m) \equiv 1013 \bmod 2026$.
If m satisfies this condition then $1013 \mid f(m)$. The numbers 1013 and 6 are relatively prime, so this divisibility is equivalent to $1013 \mid 6f(m)$, that is,

$$1013 \mid m(m+1)(2m+1).$$

It can quickly be verified that 1013 is prime, so the least positive integer m for which 1013 divides this product is $m = 506$. Computation shows that $f(506)$ is also odd, so $f(506)$ is an odd multiple of 1013 and therefore satisfies $f(506) \equiv 1013 \bmod 2026$. In conclusion, **506** is the least positive integer m satisfying the constraints of the problem.

TIEBREAKER 2

Define the sequence $\{a_n\}$ by $a_1 = 1$, and for all $k \geq 1$,

$$a_{k+1} = \left\lfloor \frac{k+1}{a_k + 1} \right\rfloor.$$

Compute a_{2026}.

Student Name:______________________

Team Name:______________________

TB-2 Answer:

TIEBREAKER 2

Define the sequence $\{a_n\}$ by $a_1 = 1$, and for all $k \geq 1$,

$$a_{k+1} = \left\lfloor \frac{k+1}{a_k + 1} \right\rfloor.$$

Compute a_{2026}.

$\boxed{1013}$ Find the first few values of a_k to find a pattern. For $k = 2$:

$$a_2 = \left\lfloor \frac{2}{a_1 + 1} \right\rfloor = \left\lfloor \frac{2}{2} \right\rfloor = 1.$$

Similarly, $a_3 = \lfloor 3/2 \rfloor = 1$, $a_4 = \lfloor 4/2 \rfloor = 2$, $a_5 = \lfloor 5/3 \rfloor = 1$, $a_6 = 3$, $a_7 = 1$, and $a_8 = 4$. It appears that $a_k = 1$ whenever k is odd, and $a_k = k/2$ whenever k is even. This can be shown using induction on k.

The discussion above proves all base cases $k \leq 8$. Now for the inductive step, assume that the statement is true for some $k = t$, where $t \geq 8$. If $t+1$ is even, t is odd, so $a_t = 1$ from the inductive hypothesis, and

$$a_{t+1} = \left\lfloor \frac{t+1}{a_t + 1} \right\rfloor = \left\lfloor \frac{t+1}{2} \right\rfloor = \frac{t+1}{2}.$$

If $t+1$ is odd, t is even, so $a_t = t/2$ from the inductive hypothesis, and

$$a_{t+1} = \left\lfloor \frac{t+1}{a_t + 1} \right\rfloor = \left\lfloor \frac{t+1}{\frac{t}{2} + 1} \right\rfloor = \left\lfloor \frac{2t+2}{t+2} \right\rfloor = 1.$$

This proves the inductive step, which completes the proof of the claim.

Now that the claim is proven, it shows that $a_{2026} = 2026/2 = \mathbf{1013}$.

TIEBREAKER 3

A square is in the coordinate plane. The product of the slopes of two sides of the square is 2026. Compute the product of the slopes of the other two sides.

Student Name:______________________

Team Name:______________________

TB-3 Answer:

TIEBREAKER 3

A square is in the coordinate plane. The product of the slopes of two sides of the square is 2026. Compute the product of the slopes of the other two sides.

$\boxed{\frac{1}{2026}}$ Let the slope of a side be m. Then the opposite side also has a slope of m. The two adjacent sides are perpendicular, so their slopes are $-\frac{1}{m}$. The product of all four sides is $m^2 \cdot \left(-\frac{1}{m}\right)^2 = 1$. If the product of the slopes of two of the sides is 2026, then the product of the slopes of the other two sides is $\mathbf{\frac{1}{2026}}$.