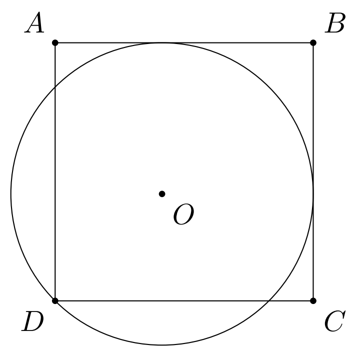
NYSML
Team Round
20 minutes -- no calculators permitted
5 points each -- 50 total points

2024 Championships *The word "compute" calls for an exact answer in simplest form.*

T1. A jar has only red and blue jellybeans. The probability of drawing a red jellybean is $\frac{4}{11}$. After two red jellybeans and half of the blue jellybeans are removed from the jar, the probability of drawing a red jellybean is $\frac{1}{2}$. Compute the number of jellybeans originally in the jar.

T2. Compute the positive integer n such that $(n-15)^3(n^2-36) = 2024$.

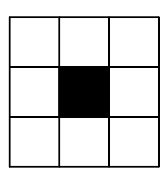
T3. Square $ABCD$ has sides $\overline{AB}$ and $\overline{BC}$ tangent to a circle centered at O with radius 6. The point D is on the circle centered at O.



Compute the area of $ABCD$.

T4. Suppose that points P and Q lie on sides $\overline{AB}$ and $\overline{CD}$ of square $ABCD$, respectively, so that $AP : PB = DQ : QC = 1 : 3$. Given that a point X is chosen uniformly at random within the square, compute the probability that $\angle PXQ$ is obtuse.

T5. Consider the following grid of squares, 8 of which are blank.



Compute the number of ways there are to place the numbers from 1 to 8 in the blank squares in the grid so that no two adjacent numbers share a common integer factor greater than 1.

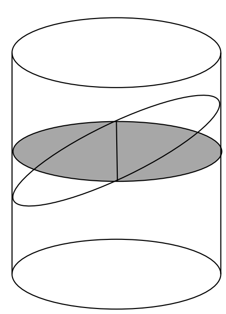
T6. The measure of an exterior angle of a regular polygon is $(2x+6)^\circ$ for some integer x. The measure of an interior angle of the same regular polygon is $(ax+b)^\circ$ where a and b are positive integers whose sum is 29. Compute the number of diagonals of the regular polygon.

T7. Let r_1, r_2, and r_3 be the three distinct complex roots of $x^3+9x^2+20x+24$. Given that r_a, r_b, and r_c are chosen uniformly at random from the set $\{r_1, r_2, r_3\}$ (with replacement), compute the mean of all 27 possible (not necessarily distinct) values of $r_a \cdot r_b + r_c$.

T8. The 17-digit number $59396\underline{A}5776\underline{B}676732$ is divisible by 308. Compute $A^2 \cdot B^3$.

T9. Compute the number of ordered triples of positive integers (a, b, c) such that
$a+b+c+\gcd(a,b,c)=18$.

T10. A right circular cylinder has a height of 10 and its base has redius 1. A triangle with its three vertices on the lateral surface of the cylinder has area 2. None of these vertices is on either of the bases of the cylinder. This triangle is inscribed in an ellipse of area $r\pi$ that has its entire boundary on the surface of the cylinder and whose minor axis is a diameter of a horizontal cross-section, as shown below, with the horizontal cross-section shaded.



Compute the least possible value of r^2.

NYSML
Team Round
20 minutes -- no calculators permitted
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2024 Championships *The word "compute" calls for an exact answer in simplest form.*

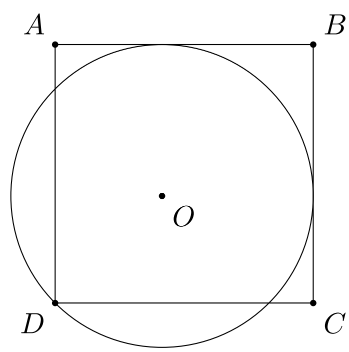
T1. A jar has only red and blue jellybeans. The probability of drawing a red jellybean is $\frac{4}{11}$. After two red jellybeans and half of the blue jellybeans are removed from the jar, the probability of drawing a red jellybean is $\frac{1}{2}$. Compute the number of jellybeans originally in the jar.

T1-Sol. $\boxed{44}$ Suppose the jar originally contained $4k$ red jellybeans, which means it contained $11k - 4k = 7k$ blue jellybeans. After the jellybeans described in the problem statement are removed, there are $4k - 2$ red jellybeans and $\frac{7k}{2}$ blue jellybeans. Because the probability of drawing a red jellybean is $\frac{1}{2}$ after the removal, it follows that $4k - 2 = \frac{7k}{2}$. This implies $8k - 4 = 7k \to k = 4$. Thus there were originally $11 \cdot 4 = \mathbf{44}$ jellybeans in the jar.

T2. Compute the positive integer n such that $(n-15)^3(n^2-36) = 2024$.

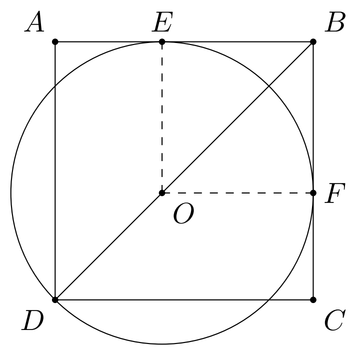
T2-Sol. $\boxed{17}$ The given equation is equivalent to $(n-15)^3(n-6)(n+6) = 2024$. Notice also that $2^3 \cdot 11 \cdot 23 = 2024$. Because these factorings both equal 2024, the factors on the left-hand side of the one equation must match up with the factors of the left-hand side of the other equation, or else one of the factors on one of the left-hand sides is 1. It can be shown that the first case is correct, and $n = \mathbf{17}$ works.

T3. Square $ABCD$ has sides $\overline{AB}$ and $\overline{BC}$ tangent to a circle centered at O with radius 6. The point D is on the circle centered at O.



Compute the area of $ABCD$.

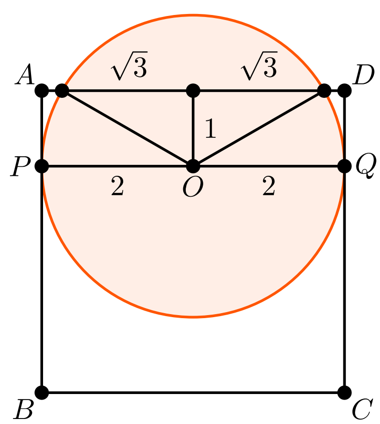
T3-Sol. $\boxed{54+36\sqrt{2}}$ Draw the two radii to the two points of tangency labeled E and F on sides $\overline{AB}$ and $\overline{BC}$, respectively.



By the equal tangents theorem, $BE = BF$. Because each radius is perpendicular to the tangent line at the point of tangency, it follows that $BEOF$ is a rectangle. Two pairs of adjacent sides are congruent ($BE = BF$ and $OE = OF$), and so $BEOF$ is a square with side length 6. Thus the diagonal $\overline{BD}$ passes through point O and its length is $BD = BO + OD = 6\sqrt{2} + 6$, so $[ABCD] = \frac{(6\sqrt{2}+6)^2}{2} = \mathbf{54 + 36\sqrt{2}}$.

T4. Suppose that points P and Q lie on sides $\overline{AB}$ and $\overline{CD}$ of square $ABCD$, respectively, so that $AP : PB = DQ : QC = 1 : 3$. Given that a point X is chosen uniformly at random within the square, compute the probability that $\angle PXQ$ is obtuse.

T4-Sol. $\boxed{\frac{\pi}{6} + \frac{\sqrt{3}}{16} \text{ or } \frac{8\pi+3\sqrt{3}}{48}}$ Without loss of generality, suppose that the side length of the square is 4. Consider the circle with diameter $\overline{PQ}$ and center O. The points X for which $\angle PXQ$ is obtuse are inside this circle.

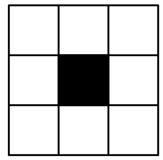


Draw the intersections the circle has with $\overline{AD}$. Notice that the distance from the center of the circle, O, to $\overline{AD}$ is 1, while the radius of the circle is 2. Thus the central angle passing through these two intersection points measures $120°$. Split the desired area into a $240°$ sector and a triangle to find that its area is

$$\frac{2}{3} \cdot 2^2 \cdot \pi + \frac{1}{2} \cdot 1 \cdot 2\sqrt{3} = \frac{8\pi}{3} + \sqrt{3}.$$

Because the area of the square is 16, it follows that the final answer is $\frac{8\pi/3+\sqrt{3}}{16} = \mathbf{\frac{8\pi+3\sqrt{3}}{48}}$.

T5. Consider the following grid of squares, 8 of which are blank.



Compute the number of ways there are to place the numbers from 1 to 8 in the blank squares in the grid so that no two adjacent numbers share a common integer factor greater than 1.

T5-Sol. $\boxed{576}$ Any two even integers share a common factor of 2, so no two even integers can be adjacent. This means that the four even integers must occupy either all the edges or all the corners. Equivalently, all four odd integers must occupy either all the corners or all the edges. First, place the odd integers into the grid so that none are adjacent. There are 2 ways to choose whether to put them all on corners or on edges, and there are 4! ways to arrange them after making this choice. Now note that any odd integer other than 3 can neighbor any even integer, so there are no restrictions on the neighbors of 1, 5, or 7. However, 3 is not allowed to be next to 6. This means that, no matter how the odd integers are placed, there are only two places that the 6 can be placed. After choosing the position of 6, there are 3! ways to place the remaining numbers 2, 4, and 8.
In conclusion, the number of ways to place these numbers under the given restriction is

$$2 \cdot 4! \cdot 2 \cdot 3! = 4! \cdot 2 \cdot 2 \cdot 3! = 24 \cdot 24 = \mathbf{576}.$$

T6. The measure of an exterior angle of a regular polygon is $(2x+6)^\circ$ for some integer x. The measure of an interior angle of the same regular polygon is $(ax+b)^\circ$ where a and b are positive integers whose sum is 29. Compute the number of diagonals of the regular polygon.

T6-Sol. $\boxed{54}$ In a polygon, because an interior angle and its corresponding exterior angle are supplementary, it follows that $2x+6+ax+b = 180 \to (2+a)x = 174-b \to x = \frac{174-b}{2+a}$. Because $a+b = 29$, it follows that $x = \frac{145+a}{2+a} = 1 + \frac{143}{2+a}$.
Thus $2+a$ is a factor of $143 = 11 \cdot 13$, and so $a = 9$ or $a = 11$.
Consider first the case that $n = 9$. This implies $b = 20$, which implies $x = \frac{154}{11} = 14$. Thus the exterior angle measures $(2 \cdot 14 + 6)^\circ = 34^\circ$, but this is not a factor of 360, so this case is impossible. Now consider the case that $n = 11$. This implies $b = 18$, which implies $x = \frac{156}{13} = 12$. Thus the exterior angle measures $(2 \cdot 12 + 6)^\circ = 30^\circ$, and the number of sides is $360 \div 30 = 12$. The number of diagonals is $\frac{12 \cdot 9}{2} = \mathbf{54}$.

T7. Let r_1, r_2, and r_3 be the three distinct complex roots of $x^3 + 9x^2 + 20x + 24$. Given that r_a, r_b, and r_c are chosen uniformly at random from the set $\{r_1, r_2, r_3\}$ (with replacement), compute the mean of all 27 possible (not necessarily distinct) values of $r_a \cdot r_b + r_c$.

T7-Sol. $\boxed{6}$ Because there are 3 choices for each root, the plan is to find the sum of all $r_a r_b + r_c$ and divide that sum by 27. Label the roots r_1, r_2, and r_3. For a given r_c, there are $3 \cdot 3 = 9$ not-necessarily-distinct values for $r_a r_b$, so the sum of all r_c is $9(r_1 + r_2 + r_3)$. If $r_a = r_b$, then there are 3 values for r_c, so the sum of all $r_a r_b$ such that $r_a = r_b$ is $3(r_1^2 + r_2^2 + r_3^2)$. If $r_a \neq r_b$, then there are still 3 values for r_c but there are 2 ways for the value of $r_a r_b$ to be achieved because of the commutative property of multiplication. The sum of all of these is $6(r_1 r_2 + r_2 r_3 + r_3 r_1)$. Thus the average is

$$\frac{3(r_1^2 + r_2^2 + r_3^2) + 6(r_1 r_2 + r_2 r_3 + r_3 r_1) + 9(r_1 + r_2 + r_3)}{27},$$

which equals

$$\frac{3((r_1 + r_2 + r_3)^2 - 2(r_1 r_2 + r_2 r_3 + r_3 r_1)) + 6(r_1 r_2 + r_2 r_3 + r_3 r_1) + 9(r_1 + r_2 + r_3)}{27},$$

which equals

$$\frac{(r_1 + r_2 + r_3)(r_1 + r_2 + r_3 + 3)}{9}.$$

This can be evaluated using Vieta's formulas. Specifically, notice that $r_1 + r_2 + r_3 = -9$, so the average is $\frac{(-9)(-9+3)}{9} = \mathbf{6}$.

Alternate Solution: If r_a, r_b, and r_c are treated as independent identically distributed random variables, then properties of expectation (linearity and independence) and Vieta's formulas drastically simplify the problem:
$\mathbb{E}[r_a r_b + r_c] = \mathbb{E}[r_a]\mathbb{E}[r_b] + \mathbb{E}[r_c] = \left(\frac{r_1+r_2+r_3}{3}\right)^2 + \frac{r_1+r_2+r_3}{3} = 9 + (-3) = 6.$

T8. The 17-digit number $59396\underline{A}5776\underline{B}676732$ is divisible by 308. Compute $A^2 \cdot B^3$.

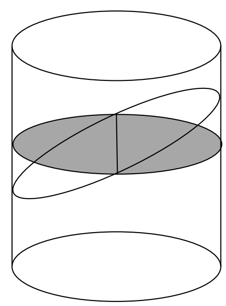
T8-Sol. $\boxed{125}$ Note that $308 = 2^2 \cdot 7 \cdot 11$ and that $1001 = 7 \cdot 11 \cdot 13$, so subtract convenient multiples of 1001 to make computations easier.
Notice that $59396\underline{A}5776\underline{B}676732 - 59359300000732732 = 37\underline{A-3}5776\underline{B-1}94400$. Subtract another $37\underline{A-3}37\underline{A-3}0000000$ to obtain $20\underline{10-A}6\underline{B-1}944000$. Subtract another 20020000000 to obtain $\underline{10-A}4\underline{B-1}944000$. This implies that $10 - A = 9 \rightarrow A = 1$ and $B - 1 = 4 \rightarrow B = 5$. The answer is $A^2 \cdot B^3 = \mathbf{125}$.

T9. Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + \gcd(a, b, c) = 18$.

T9-Sol. $\boxed{144}$ Let $d = \gcd(a, b, c)$. There exist integers a', b', and c' such that $a = d \cdot a'$, $b = d \cdot b'$,

and $c = d \cdot c'$ with $\gcd(a', b', c') = 1$. Thus it follows that $a' + b' + c' = \frac{18}{d} - 1$.
Because a', b', and c' are positive integers, it follows that d divides 18 and $a' + b' + c'$ is at least 3. There are 6 divisors of 18 but $d \neq 18$, $d \neq 9$, and $d \neq 6$ because $a' + b' + c' \geq 3$. Consider the remaining cases.
<u>Case 1:</u> Suppose $d = 3$. Then $a' + b' + c' = 5$. The only possible solutions for (a', b', c') are $(1, 1, 3)$, $(1, 2, 2)$, and each of their 3 different permutations. There are thus 6 different triples for $d = 3$.
<u>Case 2:</u> Suppose $d = 2$. Then $a' + b' + c' = 8$. Using "sticks and stones", the total number of potential solutions to this equation is $\binom{7}{2} = 21$. This includes solutions where $\gcd(a', b', c') = 2$ so these must be removed. The only solution where $\gcd(a', b', c') = 2$ is $(2, 2, 4)$ and its 3 permutations. It is not possible for $\gcd(a', b', c')$ to be greater than 2 because that would imply that $a' + b' + c' > 8$ which would be a contradiction. Thus there are $21 - 3 = 18$ triples for $d = 2$.
<u>Case 3:</u> Suppose $d = 1$. Then $a' + b' + c' = 17$. Using "sticks and stones", the total number of potential solutions to this equation is $\binom{16}{2} = 120$. Because 17 is prime, it follows that $\gcd(a', b', c') = 1$ for all solutions, so there are 120 triples for $d = 1$.
The total number of ordered triples (a, b, c) is $6 + 18 + 120 = \mathbf{144}$.

T10. A right circular cylinder has a height of 10 and its base has redius 1. A triangle with its three vertices on the lateral surface of the cylinder has area 2. None of these vertices is on either of the bases of the cylinder. This triangle is inscribed in an ellipse of area $r\pi$ that has its entire boundary on the surface of the cylinder and whose minor axis is a diameter of a horizontal cross-section, as shown below, with the horizontal cross-section shaded.



Compute the least possible value of r^2.

T10-Sol. $\boxed{\frac{64}{27} \text{ or } 2\frac{10}{27}}$ The smallest possible ellipse that lies on the lateral surface area of the cylinder is a horizontal cross-section, being a circle with radius 1. The triangle with the greatest area that can be inscribed in this circle has circumradius 1, and therefore side length $\sqrt{3}$ and area $3\sqrt{3}/4$. This area is strictly less than 2, so the ellipse is not large enough to contain the triangle in question.
Let $\mathcal{E}$ be an ellipse of minimal area circumscribing a triangle of area 2. Its semiminor axis has been given to be 1. Let the semimajor axis be d: then the area of $\mathcal{E}$ is $d \cdot 1\pi = d\pi$. The ellipse $\mathcal{E}$ is a dilation of a circle of radius 1 by a factor of d, so the area of any triangle inscribed in $\mathcal{E}$ is d times the area of a triangle inscribed in a circle of radius 1. Therefore the greatest possible area of a triangle inscribed in $\mathcal{E}$ is $d \cdot \frac{3\sqrt{3}}{4}$. The minimum possible d for which this could equal 2 is

$$d = \frac{2}{(3\sqrt{3}/4)} = \frac{8}{3\sqrt{3}}.$$

This must be the semimajor axis of $\mathcal{E}$. The area of $\mathcal{E}$ is therefore $\frac{8}{3\sqrt{3}}\pi$, so the minimal value of r is $\frac{8}{3\sqrt{3}}$ and that of r^2 is $\mathbf{\frac{64}{27}}$.

2024 Championships *Divisibility Criteria*

Remember that no calculators are allowed on this contest.
To receive full credit, the presentation must be legible, orderly, clear, and concise. When a numerical answer or formula is called for, circle or box it. Even if not completed, earlier numbered items may be used to solve later numbered items, but not vice-versa. The pages submitted should be numbered in consecutive order at the top of each page.
Put your Team Number (not your Team Name) on the cover sheet used as the first page of the papers submitted. ***Do not identify your team in any other way.***

BACKGROUND This Power Question concerns itself with *divisibility criteria*. A divisibility criterion is a test for the divisibility of a number n by a number p using the digits of n without actually performing the division of n by p. Some of these criteria may be familiar to many of our contest-takers from elementary or middle school. The goal of this Power Question is to establish some old familiar results and then move toward Zbikowski's Criterion, which is far more general than divisibility criteria that may be familiar.

P1. Some divisibility rules use just a digit or two of the number n. State and justify criteria for determining (using only a digit or two) whether a positive integer n is divisible by:
a) 5;
b) 4. *4 points*

P2. Some divisibility rules use all of the digits of the number n. State and justify criteria for determining (using all of the digits) whether a positive integer n is divisible by:
a) 9;
b) 11. *6 points*

P3. It is said that, for integers x and y, x *divides* y if $y = n \cdot x$ for some integer n.
a) Show that for all positive integers a, b, k, and m, if p divides a and p divides b, then p divides $k \cdot a + m \cdot b$.
b) Show that for all positive integers a, b, k, and m and for all primes p, if p divides a but p divides neither m nor b, then p does not divide $k \cdot a + m \cdot b$. *5 points*

P4. A common divisibility rule for 7 may be somewhat familiar, and it is similar to a rule for divisibility by 17.
a) Show that if $n = 10a + b$ for nonnegative integers a and b, n is divisible by 7 if and only if $a - 2b$ is divisible by 7. You may find it helpful to write $10a + b$ as $10(a - 2b) + 21b$.
b) Show that if $n = 10a + b$ for nonnegative integers a and b, n is divisible by 17 if and only if $a - 5b$ is divisible by 17. *5 points*

P5. This question involves finding criteria for divisibility that are similar to the ones in **P-4**.
a) Find an integer k_1 with $|k_1| < 19$ such that the following statement is true: If $n = 10a + b$ for nonnegative integers a and b, then n is divisible by 19 if and only if $a - k_1 b$ is divisible by 19.
b) Find an integer k_2 with $|k_2| < 23$ such that the following statement is true: If $n = 10a + b$ for nonnegative integers a and b, then n is divisible by 23 if and only if $a - k_2 b$ is divisible by 23.
5 points

P6. The remainder of this Power Question will focus on Zbikowski's Criterion for divisibility. Zbikowski originally published his work in a journal in 1861, and the criterion appears to have been previously unpublished.
There is a "subtraction style Zbikowski test" for a divisor p that operates as follows: Find a multiple of p that ends in 1. Truncate (i.e., remove) the 1 from this multiple and let k denote the integer that results. Then it follows that $n = 10a + b$ (for nonnegative integers a and b) is divisible by p if and only if $a - kb$ is divisible by p.
a) Show that for there to be a multiple of p that ends in 1, p must be relatively prime to 10 (that is, p must end in 1, 3, 7, or 9).
b) Justify that the subtraction style Zbikowski test is valid. *6 points*

P7. There is an "addition style Zbikowski test" for a divisor p that operates as follows: Find a multiple of p that ends in 9. Truncate the 9 from this multiple and let k denote 1 more than the integer that results. Then it follows that $n = 10a + b$ is divisible by p if and only if $a + kb$ is divisible by p. Justify that the addition style Zbikowski test is valid. *4 points*

P8. Find and justify divisibility criteria for the primes 53 and 2027 using the subtraction style Zbikowski test or the addition style Zbikowski test. *6 points*

P9. Suppose that p divides a positive integer of the form $10k - 1$. Prove that if $n = \underline{a_m}\,\underline{a_{m-1}}\ \ldots\ \underline{a_1}\,\underline{a_0}$, then p divides n if and only if p divides $a_m + k(a_{m-1} + k(a_{m-2} + \cdots + k(a_1 + ka_0)\cdots))$. *4 points*

P10. What if n is written in base 8 (octal) or base 16 (hexadecimal)? What changes about Zbikowski's tests? What stays the same? *5 points*

NYSML

Power Question

60 minutes -- no calculators permitted

50 total points

2024 Championships *Divisibility Criteria*

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P1. Some divisibility rules use just a digit or two of the number n. State and justify criteria for determining (using only a digit or two) whether a positive integer n is divisible by:
a) 5;
b) 4. *4 points*

P1-Sol. **a)** A rule for divisibility by 5 is that n is divisible by 5 if and only if its ones digit is divisible by 5 (that is, if the last digit is 0 or 5). Note that $n = 10a + b$ for some positive integers a and b. Suppose that b is divisible by 5. Then $b = 5c \to n = 5(2a + c)$, and because the integers are closed under addition and multiplication, $2a + c$ is an integer, so n is divisible by 5. Conversely, if b is not a multiple of 5, then $b = 5c + d$ for digits c and d where $1 \leq d \leq 4$. Thus $n = 5(2a + c) + d$, which is not a multiple of 5.

b) A rule for divisibility by 4 is that n is divisible by 4 if and only if the two-digit number formed by its tens digit and ones digit is divisible by 4. Note that $n = 100a + b$ for some positive integers a and b. Suppose that b is divisible by 4. Then $b = 4c \to n = 4(25a + c)$, and because the integers are closed under addition and multiplication, $25a + c$ is an integer, so n is divisible by 4. Conversely, if b is not a multiple of 4, then $b = 4c + d$ for whole numbers c and d where $d \in \{1, 2, 3\}$. Thus $n = 4(25a + c) + d$, which is not a multiple of 4.

P2. Some divisibility rules use all of the digits of the number n. State and justify criteria for determining (using all of the digits) whether a positive integer n is divisible by:
a) 9;
b) 11. *6 points*

P2-Sol. a) A rule for divisibility by 9 is that n is divisible by 9 if and only if the digit sum of n is divisible by 9. Let $n = \underline{n_k}\ \dots\ \underline{n_4}\,\underline{n_3}\,\underline{n_2}\,\underline{n_1}$. Then it follows that
$n = (999\dots9)n_k + \dots + 999n_4 + 99n_3 + 9n_2) + (n_k + \dots + n_4 + n_3 + n_2 + n_1)$. If the sum of the digits is divisible by 9, then that digit sum equals $9 \cdot c$ for some positive integer c, and
$n = 9((111\dots1)n_k + \dots + 111n_4 + 11n_3 + n_2 + c)$, and because the integers are closed under addition and multiplication, $(111\dots1)n_k + \dots + 111n_4 + 11n_3 + n_2 + c$ is an integer, so n is divisible by 9. Conversely, if the digit sum is not a multiple of 9, a similar argument will show that n is not divisible by 9.

b) A rule for divisibility by 11 is that n is divisible by 11 if and only if the "alternating digit sum" of n is divisible by 11. Let $n = \underline{n_k}\ \dots\ \underline{n_4}\,\underline{n_3}\,\underline{n_2}\,\underline{n_1}$. Then it follows that
$n = (10^{k-1} + (-1)^k)n_k + \dots + 1001n_4 + 99n_3 + 11n_2) + ((-1)^{k-1}n_k + \dots - n_4 + n_3 - n_2 + n_1)$. If the alternating sum of the digits is divisible by 11, then that digit sum equals $11 \cdot c$ for some positive integer c, and because every coefficient of the n_i in
$(10^{k-1} + (-1)^k)n_k + \dots + 1001n_4 + 99n_3 + 11n_2$ is a multiple of 11, and because the integers are closed under addition and multiplication, it follows that n is divisible by 11. Conversely, if the alternating digit sum is not a multiple of 11, a similar argument will show that n is not divisible by 11.

P3. It is said that, for integers x and y, x *divides* y if $y = n \cdot x$ for some integer n.
a) Show that for all positive integers a, b, k, and m, if p divides a and p divides b, then p divides $k \cdot a + m \cdot b$.
b) Show that for all positive integers a, b, k, and m and for all primes p, if p divides a but p divides neither m nor b, then p does not divide $k \cdot a + m \cdot b$. *5 points*

P3-Sol. a) There exist positive integers c and d such that $a = p \cdot c$ and $b = p \cdot d$. Notice that $k \cdot a + m \cdot b = k \cdot p \cdot c + m \cdot p \cdot d = p(k \cdot c + m \cdot d)$. Because the integers are closed under addition and multiplication, it follows that $k \cdot a + m \cdot b$ is a multiple of p, as needed.
b) Assume for the sake of contradiction that p does divide $k \cdot a + m \cdot b$. This implies $k \cdot a + m \cdot b = n \cdot p$. Because p divides a, it follows that $a = y \cdot p$ for some integer y, and thus $m \cdot b = n \cdot p - ky \cdot p = (n - ky) \cdot p$. Because the integers are closed under addition and multiplication, this implies p divides mb, which implies p divides either m or b, and this is a contradiction.

P4. A common divisibility rule for 7 may be somewhat familiar, and it is similar to a rule for divisibility by 17.
a) Show that if $n = 10a + b$ for nonnegative integers a and b, n is divisible by 7 if and only if $a - 2b$ is divisible by 7. You may find it helpful to write $10a + b$ as $10(a - 2b) + 21b$.
b) Show that if $n = 10a + b$ for nonnegative integers a and b, n is divisible by 17 if and only if $a - 5b$ is divisible by 17. *5 points*

P4-Sol. a) Note that $10a + b = 10(a - 2b) + 21b$. Suppose that $a - 2b$ is a multiple of 7. Because $21b$ is also a multiple of 7, it follows by **P-3** that n is a multiple of 7. Conversely, if $a - 2b$ is not a multiple of 7, then $a - 2b = 7k + r$ for some whole numbers k and r where $1 \leq r \leq 6$. Thus it follows that $n = 10 \cdot 7k + 10 \cdot r + 21b = 7(10k + 3b) + 10r$, and because $10r$ is not a multiple of 7, n is not a multiple of 7.

b) Note that $10a + b = 10(a - 5b) + 51b$. Suppose that $a - 5b$ is a multiple of 17. Because $51b$ is also a multiple of 17, it follows by **P-3** that n is a multiple of 17. Conversely, if $a - 5b$ is not a multiple of 17, then $a - 5b = 17k + r$ for some whole numbers k and r where $1 \leq r \leq 16$. Thus it follows that $n = 10 \cdot 17k + 10 \cdot r + 51b = 17(10k + 3b) + 10r$, and because $10r$ is not a multiple of 17, n is not a multiple of 17.

P5. This question involves finding criteria for divisibility that are similar to the ones in **P-4**.
a) Find an integer k_1 with $|k_1| < 19$ such that the following statement is true: If $n = 10a + b$ for nonnegative integers a and b, then n is divisible by 19 if and only if $a - k_1 b$ is divisible by 19.
b) Find an integer k_2 with $|k_2| < 23$ such that the following statement is true: If $n = 10a + b$ for nonnegative integers a and b, then n is divisible by 23 if and only if $a - k_2 b$ is divisible by 23.
5 points

P5-Sol. a) A rule for divisibility by 19 is that $n = 10a + b$ is divisible by 19 if and only if $a + 2b$ is divisible by 19. This can be seen by writing $n = 10a + b = 10(a + 2b) - 19b$ and applying an argument similar to the argument made in the solution to **P-4**.

b) A rule for divisibility by 23 is that $n = 10a + b$ is divisible by 23 if and only if $a + 7b$ is divisible by 23. This can be seen by writing $n = 10a + b = 10(a + 7b) - 69b$ and applying an argument similar to the argument made in the solution to **P-4**.

P6. The remainder of this Power Question will focus on Zbikowski's Criterion for divisibility. Zbikowski originally published his work in a journal in 1861, and the criterion appears to have been previously unpublished.
There is a "subtraction style Zbikowski test" for a divisor p that operates as follows: Find a multiple of p that ends in 1. Truncate (i.e., remove) the 1 from this multiple and let k denote the integer that results. Then it follows that $n = 10a + b$ (for nonnegative integers a and b) is divisible by p if and only if $a - kb$ is divisible by p.
a) Show that for there to be a multiple of p that ends in 1, p must be relatively prime to 10 (that is, p must end in 1, 3, 7, or 9).
b) Justify that the subtraction style Zbikowski test is valid. *6 points*

P6-Sol. a) If p is even, no multiple of p can end in 1 because then that multiple would be odd. Similarly, no multiple of 5 can end in 1 by the divisibility test for 5. Thus the units digit of p must be 1, 3, 7, or 9.

b) Write $n = 10a + b = 10(a - kb) + (10k + 1)b$. By construction, $10k + 1$ is a multiple of p. Thus it follows that if $a - kb$ is a multiple of p, by **P-3** n is a multiple of p. Similarly, if $a - kb$ is not a multiple of p, by an argument similar to the one in the solution to **P-3** n is not a multiple of p.

P7. There is an "addition style Zbikowski test" for a divisor p that operates as follows: Find a multiple of p that ends in 9. Truncate the 9 from this multiple and let k denote 1 more than the integer that results. Then it follows that $n = 10a + b$ is divisible by p if and only if $a + kb$ is divisible by p. Justify that the addition style Zbikowski test is valid. *4 points*

P7-Sol. Write $n = 10a + b = 10(a + kb) - (10k - 1)b$. By construction, $10k - 1$ is a multiple of p. Thus it follows that if $a + kb$ is a multiple of p, by **P-3** n is a multiple of p. Similarly, if $a + kb$ is not a multiple of p, by an argument similar to the one in the solution to **P-3** n is not a multiple of p.

P8. Find and justify divisibility criteria for the primes 53 and 2027 using the subtraction style Zbikowski test or the addition style Zbikowski test. *6 points*

P8-Sol. For the prime 53, a divisibility criterion would be that $n = 10a + b$ is divisible by 53 if and only if $a + 16b$ is divisible by 53. This is because of the addition style Zbikowski test from **P-7**. Note that 159 is a multiple of 53, and $15 + 1 = 16$ is the k from that test. (There are many similar possible divisibility criteria.)
For the prime 2027, use the subtraction style Zbikowski test from **P-6**. A divisibility criterion might be that $n = 10a + b$ is divisible by 2027 if and only if $a - 608b$ is divisible by 2027. Note that $6081 = 2027 \cdot 3$ is a multiple of 2027, and 608 is the k from that test. (There are many similar possible divisibility criteria.)

P9. Suppose that p divides a positive integer of the form $10k - 1$. Prove that if $n = \underline{a_m}\,\underline{a_{m-1}}\,\ldots\,\underline{a_1}\,\underline{a_0}$, then p divides n if and only if p divides $a_m + k(a_{m-1} + k(a_{m-2} + \cdots + k(a_1 + ka_0)\cdots))$. *4 points*

P9-Sol. Note that if p divides $10k - 1$, then $10k \equiv 1 \pmod{p}$. Notice that $n = 10(10^{m-1}a_m + \cdots + 10a_2 + a_1) + a_0$, which is equivalent to $10(10^{m-1}a_m + \cdots + 10a_2 + a_1) + 10ka_0 = 10(10^{m-1}a_m + \cdots + 10a_2 + (a_1 + ka_0) \pmod{p}$. This can be rewritten as $n \equiv 10^2(10^{m-2}a_m + \cdots + 10a_3 + a_2) + 10 \cdot 10k(a_1 + ka_0) \pmod{p}$. Continuing in this way, it follows that $n \equiv 10^m(a_m + k(a_{m-1} + k(a_{m-2} + \cdots + k(a_1 + ka_0)\cdots)) \pmod{p}$. Because p divides $10k - 1$, it follows that neither 2 nor 5 is a factor of p, and so p does not divide

10^m. Thus p divides n if and only if p divides $a_m + k(a_{m-1} + k(a_{m-2} + \cdots + k(a_1 + ka_0)\cdots))$.

P10. What if n is written in base 8 (octal) or base 16 (hexadecimal)? What changes about Zbikowski's tests? What stays the same? *5 points*

P10-Sol. If n is written as an octal number, then the fundamental ideas remain the same regarding how to construct divisibility criteria, but some differences appear. For example, when constructing an addition style Zbikowski test, one would look for a multiple of p that ends in 7 (the greatest digit in base eight). When constructing a subtraction style Zbikowski test, one would still look for a multiple of p that ends in 1 (the least positive digit). Similar ideas are present with hexadecimal representations. What comparisons and contrasts has the reader drawn? Send them to `coachreu@gmail.com` and we'll try to compile several of them for our next NYSML book.

You can find more information about Zbikowski criteria at *maa.org/press/periodicals/convergence/divisibility-tests-a-history-and-users-guide-zbikowski-divisibility-tests. The main idea for this Power Question came from an article by Yonah Cherniavsky and Artour Mouftakhov (2014) Zbikowski's Divisibility Criterion, The College Mathematics Journal, 45:1, 17-21, DOI: 10.4169/college.math.j.45.1.017. NYSML also suggests the article by Sandy Ganzell (2017) Divisibility Tests, Old and New, The College Mathematics Journal, 48:1, 36-40, DOI: 10.4169/college.math.j.48.1.36.*

NYSML
Individual Round
10 min/pair -- no calculators permitted
1 point each -- 150 total points

2024 Championships *The word "compute" calls for an exact answer in simplest form.*

I1. Compute $1397^2 + 1391$.

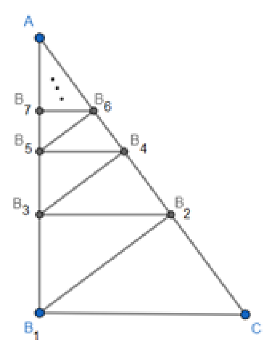
I2. When numbering the floors of a new high-rise building, the owner decides that any number that is a multiple of 3 or a multiple of 5 or both will not be used. Because of this decision, the first four floors are numbered 1, 2, 4, and 7. Given that the high-rise building has 80 floors, compute the number that will be assigned to the top floor.

I3. Consider the following system of equations:

$$\begin{aligned} 17 &= a, \\ 103 &= 2a + 3b, \\ 525 &= 4a + 5b + 6c, \\ 616 &= 7a + 8b + 9c + 10d, \\ 1224 &= 11a + 12b + 13c + 14d + 15e. \end{aligned}$$

This system has exactly one solution. For this solution, compute $a + b + c + d + e$.

I4. Given $\triangle AB_1C$ with $AB_1 = 4$, $B_1C = 3$, and $AC = 5$. Starting at B_1, a path is drawn by first dropping a perpendicular $\overline{B_1B_2}$ to $\overline{AC}$, then dropping a perpendicular $\overline{B_2B_3}$ to $\overline{AB_1}$, then dropping a perpendicular $\overline{B_3B_4}$ to $\overline{AC}$, and so on *ad infinitum*, alternately dropping perpendiculars to $\overline{AC}$ and $\overline{AB_1}$.



Compute the total length of the path $B_1B_2B_3\ldots$.

I5. Compute the positive integer n such that $(\log_2 108)^2 + (\log_2 162)^2 - 1 = (\log_2 n)^2$.

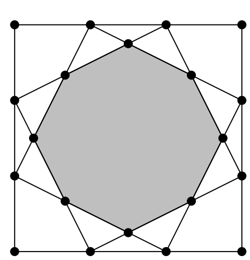
I6. In the NYSMLottery, the winning combination is determined by picking a subset of 6 numbers from the set $\{1, 2, 3, \ldots, N\}$. Given that there are 28,989,675 possible winning combinations, compute N.

I7. In the equation $x^3 + bx + c = 0$, each of b and c is independently and uniformly randomly chosen from the interval $[-2024, 2024]$. Compute the probability that all three solutions of this cubic equation are negative real numbers.

I8. Compute the least positive integer $n > 45$ such that $n^2 - 1$ is a multiple of 2024.

I9. Compute the sum of all positive three-digit multiples of 7 whose digit sum is also 7.

I10. In a square of side length 36, two squares are inscribed, with vertices at the trisection points of the sides, as shown in the diagram.



Compute the area of the intersection of these two squares, which is shaded in the diagram.

2024 Championships

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2024 Championships

I1)

I2)

Student Name:____________________ **Team Name:**_______________

2024 Championships

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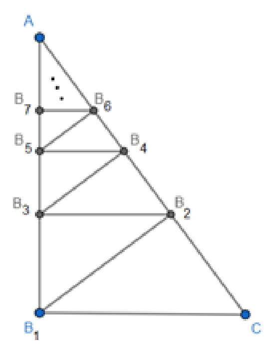
$$\begin{aligned} 17 &= a, \\ 103 &= 2a + 3b, \\ 525 &= 4a + 5b + 6c, \\ 616 &= 7a + 8b + 9c + 10d, \\ 1224 &= 11a + 12b + 13c + 14d + 15e. \end{aligned}$$

This system has exactly one solution. For this solution, compute $a + b + c + d + e$.

I4. Given $\triangle AB_1C$ with $AB_1 = 4$, $B_1C = 3$, and $AC = 5$. Starting at B_1, a path is drawn by first dropping a perpendicular $\overline{B_1B_2}$ to $\overline{AC}$, then dropping a perpendicular $\overline{B_2B_3}$ to $\overline{AB_1}$, then dropping a perpendicular $\overline{B_3B_4}$ to $\overline{AC}$, and so on *ad infinitum*, alternately dropping perpendiculars to $\overline{AC}$ and $\overline{AB_1}$.



Compute the total length of the path $B_1B_2B_3\ldots$.

2024 Championships

I3)

I4)

Student Name:____________________ Team Name:________________

2024 Championships

I5. Compute the positive integer n such that $(\log_2 108)^2 + (\log_2 162)^2 - 1 = (\log_2 n)^2$.

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2024 Championships

I5)

I6)

Student Name:______________________ **Team Name:**_________________

2024 Championships

I7. In the equation $x^3 + bx + c = 0$, each of b and c is independently and uniformly randomly chosen from the interval $[-2024, 2024]$. Compute the probability that all three solutions of this cubic equation are negative real numbers.

I8. Compute the least positive integer $n > 45$ such that $n^2 - 1$ is a multiple of 2024.

2024 Championships

I7)

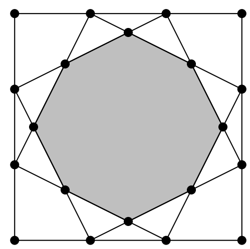
I8)

Student Name:____________________ **Team Name:**________________

2024 Championships

I9. Compute the sum of all positive three-digit multiples of 7 whose digit sum is also 7.

I10. In a square of side length 36, two squares are inscribed, with vertices at the trisection points of the sides, as shown in the diagram.



Compute the area of the intersection of these two squares, which is shaded in the diagram.

NYSML Individual Round

10 min/pair -- no calculator permitted

1 point each -- 150 total points

The word "compute" calls for an exact answer in simplest form.

2024 Championships

I9)

I10)

Student Name:______________________ **Team Name:**_________________

NYSML
Individual Round
10 min/pair -- no calculators permitted
1 point each -- 150 total points

2024 Championships *The word "compute" calls for an exact answer in simplest form.*

I1. Compute $1397^2 + 1391$.

I1-Sol. $\boxed{1953000}$ Let $N = 1400$. Then the desired quantity is $(N-3)^2 + N - 9 = N^2 - 5N$. Substituting $N = 1400$, this expression equals $1400^2 - 5 \cdot 1400 = 1960000 - 7000 = \mathbf{1953000}$.

I2. When numbering the floors of a new high-rise building, the owner decides that any number that is a multiple of 3 or a multiple of 5 or both will not be used. Because of this decision, the first four floors are numbered 1, 2, 4, and 7. Given that the high-rise building has 80 floors, compute the number that will be assigned to the top floor.

I2-Sol. $\boxed{149}$ In the first 15 floors of the high-rise building, the numbers 3, 5, 6, 9, 10, 12, and 15 will not be used. This pattern continues in blocks of 15; only 8 numbers will be used in any block of 15. Because there are 80 floors, exactly 10 groups of 8 numbers will be used. Thus the top floor will get the last number in the tenth group, which will be **149**.

I3. Consider the following system of equations:

$$\begin{aligned} 17 &= a, \\ 103 &= 2a + 3b, \\ 525 &= 4a + 5b + 6c, \\ 616 &= 7a + 8b + 9c + 10d, \\ 1224 &= 11a + 12b + 13c + 14d + 15e. \end{aligned}$$

This system has exactly one solution. For this solution, compute $a + b + c + d + e$.

I3-Sol. $\boxed{97}$ The quantity $a + b$ is found by using the first two equations:

$$3a + 3b = a + (2a + 3b) = 120 \quad \Longrightarrow \quad a + b = \frac{120}{3} = 40.$$

This method generalizes to find the partial sums of $a+b+c+d+e$ in turn.

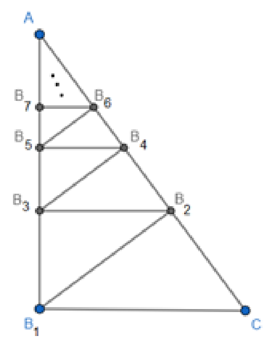
$$a+b+c = \frac{a+(a+b)+(4a+5b+6c)}{6} = \frac{17+40+525}{6} = 97,$$

$$a+b+c+d = \frac{17+40+97+616}{10} = 77,$$

$$a+b+c+d+e = \frac{17+40+97+77+1224}{15} = \mathbf{97}.$$

Note: This implies that $(a,b,c,d,e) = (17, 23, 57, -20, 20)$, which can be manually confirmed to satisfy the system of equations.

I4. Given $\triangle AB_1C$ with $AB_1 = 4$, $B_1C = 3$, and $AC = 5$. Starting at B_1, a path is drawn by first dropping a perpendicular $\overline{B_1B_2}$ to $\overline{AC}$, then dropping a perpendicular $\overline{B_2B_3}$ to $\overline{AB_1}$, then dropping a perpendicular $\overline{B_3B_4}$ to $\overline{AC}$, and so on *ad infinitum*, alternately dropping perpendiculars to $\overline{AC}$ and $\overline{AB_1}$.



Compute the total length of the path $B_1B_2B_3\ldots$.

I4-Sol. $\boxed{12}$ Because $\overline{B_1B_2}$ is the altitude of a right triangle, it follows by similar triangles that $\frac{B_1B_2}{4} = \frac{3}{5} \rightarrow B_1B_2 = \frac{12}{5}$. Note that $\triangle AB_1B_2 \sim \triangle ACB_1$ with scale factor $\frac{4}{5}$. Each successive $\triangle AB_kB_{k+1}$ will be similar to the previous one with the same scale factor, so the lengths of the segments of the path will form an infinite geometric sequence with $a = \frac{12}{5}$ and $r = \frac{4}{5}$. The sum of the terms of this sequence is $\frac{12/5}{1-4/5} = \mathbf{12}$.

I5. Compute the positive integer n such that $(\log_2 108)^2 + (\log_2 162)^2 - 1 = (\log_2 n)^2$.

I5-Sol. $\boxed{972}$ Let $x = \log_2 3$. Then

$$\log_2 108 = \log_2\left(2^2 \cdot 3^3\right) = 3\log_2 3 + 2\log_2 2 = 3x+2 \ \text{ and}$$
$$\log_2 162 = \log_2\left(2 \cdot 3^4\right) = 4\log_2 3 + \log_2 2 = 4x+1.$$

Thus the left-hand side of the given equation is equal to

$$(3x+2)^2+(4x+1)^2-1=(9x^2+12x+4)+(16x^2+8x+1)-1=25x^2+20x+4=(5x+2)^2,$$

so $\log_2 n = 5x+2$. This implies

$$\log_2 n = 5\log_2 3 + 2 = \log_2\left(2^2\cdot 3^5\right) = \log_2 972,$$

so $n = \mathbf{972}$.

I6. In the NYSMLottery, the winning combination is determined by picking a subset of 6 numbers from the set $\{1,2,3,\ldots,N\}$. Given that there are 28,989,675 possible winning combinations, compute N.

I6-Sol. $\boxed{55}$ Factor 28,989,675 $= 25\cdot 9\cdot 11\cdot 13\cdot 17\cdot 53$. Note in particular that there is no factor of 7 in 28,989,675. Note also that this number is equal to $\binom{N}{6} = \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)}{6!}$. One of the factors of the numerator of this fraction is 53, and none of these factors is a multiple of 7. The only N that allows this to happen is $N = \mathbf{55}$.

Alternate Solution: Because $\binom{N}{6}$ is a multiple of 25, it follows that $N(N-1)(N-2)(N-3)(N-4)(N-5)$ is a multiple of 125. Note that if $N \geq 125$, the number of possible winning combinations is much too large. Thus N and $N-5$ must be multiples of 5. Therefore look for N that are either multiples of 25 or 5 more than a multiple of 25. Because none of $\{N, N-1, N-2, N-3, N-4, N-5\}$ is a multiple of 7, it follows that N cannot be 25 or 30 or 50, but N could be 55, and that can be shown to be the answer.

I7. In the equation $x^3+bx+c=0$, each of b and c is independently and uniformly randomly chosen from the interval $[-2024, 2024]$. Compute the probability that all three solutions of this cubic equation are negative real numbers.

I7-Sol. $\boxed{0}$ By Vieta's formulas, the sum of the roots is 0. Thus it is impossible for all three solutions to be negative, and the desired probability is $\mathbf{0}$.

I8. Compute the least positive integer $n > 45$ such that n^2-1 is a multiple of 2024.

I8-Sol. $\boxed{461}$ Factor $2024 = 45^2-1 = 44\cdot 46 = 2^3\cdot 11\cdot 23$. In order for 8 to divide n^2-1, it is necessary and sufficient to have n be odd. Because $11 \mid n^2-1$ and $23 \mid n^2-1$, it follows that $n \equiv \pm 1 \pmod{11}$ and $n \equiv \pm 1 \pmod{23}$. Consider these four cases in light of the Chinese Remainder Theorem. If $n \equiv 1 \pmod{11}$ and $n \equiv 1 \pmod{23}$, then $n \equiv 1 \pmod{253}$. If $n \equiv -1$

(mod 11) and $n \equiv -1 \pmod{23}$, then $n \equiv -1 \equiv 252 \pmod{253}$. If $n \equiv 1 \pmod{11}$ and $n \equiv -1 \pmod{23}$, then $n \equiv 45 \pmod{253}$. (This can be seen by trial and error or by the fact that $n = 45$ satisfies $2024 \mid n^2 - 1$.) If $n \equiv -1 \pmod{11}$ and $n \equiv 1 \pmod{23}$, then $n \equiv -45 \equiv 208 \pmod{253}$. (This can be seen by trial and error or by the fact that $n = -45$ is clearly a number such that $2024 \mid n^2 - 1$.) It follows that the following numbers are the least positive integers such that $253 \mid n^2 - 1$:

$$1, 45, 208, 252, 254, 298, 461, 505, \ldots.$$

The least odd value in the above list greater than 45 is $n = \mathbf{461}$.

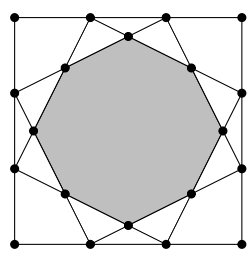
I9. Compute the sum of all positive three-digit multiples of 7 whose digit sum is also 7.

I9-Sol. $\boxed{1666}$ Suppose the number is $\underline{A}\,\underline{B}\,\underline{C}$. The problem statement implies $7 \mid 100A + 10B + C$ and $7 = A + B + C$. The first statement implies $7 \mid 2A + 3B + C$, and because $7 = A + B + C$, it follows that

$$7 \mid (2A + 3B + C) - 2(A + B + C) = B - C.$$

Because $0 \leq B, C \leq 6$, it follows that $B = C$. This means the possible numbers are 700, 511, 322, and 133, which can all be verified to work, and the sum of these is **1666**.

I10. In a square of side length 36, two squares are inscribed, with vertices at the trisection points of the sides, as shown in the diagram.



Compute the area of the intersection of these two squares, which is shaded in the diagram.

I10-Sol. $\boxed{600}$ Consider the square in the coordinate plane and let its vertices be $(0, 0)$, $(36, 0)$, $(36, 36)$, and $(0, 36)$. The lines have equations $y = -2x + 24$, $y = -\frac{1}{2}x + 12$, $y = \frac{1}{2}x + 24$, $y = 2x + 12$, and so on. The vertices of the octagon are $\{(8, 8), (3, 16), (8, 24), (18, 33), (28, 28), (33, 16), (28, 8), (18, 3)\}$. The area of the octagon is the area of one of the inner squares minus the areas of 4 small congruent right triangles: $\left(12\sqrt{5}\right)^2 - 4\left(\frac{1}{2}\right) \cdot 3\sqrt{5} \cdot 4\sqrt{5} = \mathbf{600}$.

NYSML
Relay Round
no calculators

Each Round: sub-teams of 3 -- 5 points at 3 minutes, 3 points at 6 minutes

2024 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. Compute the least positive integer greater than 2024 whose digit sum is a power of 2, and each of whose digits is a power of 2.

R1-2. *Let N be the number you will receive.* Given that the parabola with equation $y = x^2 + Ax + B$ passes through $(-1, K)$, $(0, 2024)$, and $(1, N)$, compute K.

R1-3. *Let N be the number you will receive.* Suppose $ABCD$ is a square with area N. Let the midpoints of $\overline{CD}$ and $\overline{DA}$ be M and K, respectively, and let $\overline{BM}$ and $\overline{CK}$ intersect at P. Compute AP.

R2-1. Complex numbers x and y satisfy the system of equations

$$\begin{cases} x + 3y &= 8, \\ xy &= 10. \end{cases}$$

Compute $x^2 + 9y^2$.

R2-2. *Let N be the number you will receive.* Compute

$$\frac{N^3 + 4 \cdot (N^2 \cdot 3) + 4 \cdot (N \cdot 3^2) + 3^3}{N + 3}.$$

R2-3. *Let N be the number you will receive.* Compute

$$\sqrt{\frac{N^2 - 36^2 - 48^2}{37^2 - 21^2 - 28^2}}.$$

R1-1. Compute the least positive integer greater than 2024 whose digit sum is a power of 2, and each of whose digits is a power of 2.

R1-2. *Let N be the number you will receive.* Given that the parabola with equation $y = x^2 + Ax + B$ passes through $(-1, K)$, $(0, 2024)$, and $(1, N)$, compute K.

R1-3. *Let N be the number you will receive.* Suppose $ABCD$ is a square with area N. Let the midpoints of $\overline{CD}$ and $\overline{DA}$ be M and K, respectively, and let $\overline{BM}$ and $\overline{CK}$ intersect at P. Compute AP.

R2-1. Complex numbers x and y satisfy the system of equations

$$\begin{cases} x + 3y &= 8, \\ xy &= 10. \end{cases}$$

Compute $x^2 + 9y^2$.

R2-2. *Let N be the number you will receive.* Compute

$$\frac{N^3 + 4 \cdot (N^2 \cdot 3) + 4 \cdot (N \cdot 3^2) + 3^3}{N+3}.$$

R2-3. *Let N be the number you will receive.* Compute

$$\sqrt{\frac{N^2 - 36^2 - 48^2}{37^2 - 21^2 - 28^2}}.$$

NYSML

Relay Round

no calculators

Each Round: sub-teams of 3 -- 5 points at 3 minutes, 3 points at 6 minutes

2024 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. Compute the least positive integer greater than 2024 whose digit sum is a power of 2, and each of whose digits is a power of 2.

R1-1-Sol. $\boxed{2114}$ Because each digit is a power of 2, each digit is positive. Thus the sum of the digits is at least $2+1+1+1=5$. The next greatest power of 2 is 8, so try to make the digit sum equal to 8. To minimize the second and third digits, notice that **2114** satisfies the conditions of the problem.

R1-2. *Let N be the number you will receive.* Given that the parabola with equation $y = x^2 + Ax + B$ passes through $(-1, K)$, $(0, 2024)$, and $(1, N)$, compute K.

R1-2-Sol. $\boxed{1936}$ Because the parabola passes through $(0, 2024)$, it follows that $B = 2024$. Substitute $x = 1$ to obtain

$$N = 2025 + A \implies A = N - 2025.$$

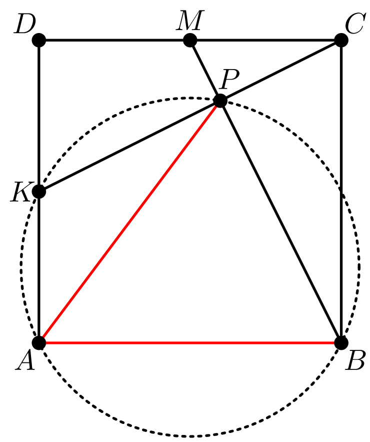
Now substitute $x = -1$ to obtain that

$$K = 2025 - A = 4050 - N.$$

With $N = 2114$, the answer is $K = \mathbf{1936}$.
Alternatively, with $N = 2114$, note that $2024 = 45^2 - 1$ and $2114 = 46^2 - 2$, so the polynomial is $(x+45)^2 - (x+1)$, giving $K = 44^2 = 1936$.

R1-3. *Let N be the number you will receive.* Suppose $ABCD$ is a square with area N. Let the midpoints of $\overline{CD}$ and $\overline{DA}$ be M and K, respectively, and let $\overline{BM}$ and $\overline{CK}$ intersect at P. Compute AP.

R1-3-Sol. $\boxed{44}$ This problem can be solved relatively cleanly with coordinates. The following solution, by contrast, is more geometric and is presented for your consideration.



Note that $\overline{BM}$ and $\overline{CK}$ are perpendicular to each other because $m\angle PCB = 90^\circ - m\angle DCK = 90^\circ - m\angle CBM$, so $m\angle BPK = 90^\circ$. Notice also that $m\angle BAK = 90^\circ$, so quadrialteral $BAKP$ is cyclic.

Because K is the midpoint of $\overline{AD}$, it follows that $m\angle CKD = m\angle BKA$, so $\overline{KA}$ bisects the supplement of $\angle PKB$. This implies that $AP = AB$. Because the area of the square is N, this means $AP = AB = \sqrt{N}$. With $N = 1936$, the answer is $AP = \mathbf{44}$.

R2-1. Complex numbers x and y satisfy the system of equations

$$\begin{cases} x + 3y &= 8, \\ xy &= 10. \end{cases}$$

Compute $x^2 + 9y^2$.

R2-1-Sol. $\boxed{4}$ Square the first equation to obtain $(x+3y)^2 = x^2 + 6xy + 9y^2 = 64$. Subtract $6xy = 60$ from both sides to obtain $x^2 + 9y^2 = \boxed{4}$.

R2-2. *Let N be the number you will receive.* Compute

$$\frac{N^3 + 4 \cdot (N^2 \cdot 3) + 4 \cdot (N \cdot 3^2) + 3^3}{N+3}.$$

R2-2-Sol. $\boxed{61}$ The numerator can be split into

$$\begin{aligned} N^3 + 4 \cdot (N^2 \cdot 3) + 4 \cdot (N \cdot 3^2) + 3^3 &= [N^3 + 3 \cdot (N^2 \cdot 3) + 3 \cdot (N \cdot 3^2) + 3^3] + N^2 \cdot 3 + N \cdot 3^2 \\ &= (N+3)^3 + 3N(N+3). \end{aligned}$$

Therefore the given fraction simplifies to

$$\frac{N^3 + 4 \cdot (N^2 \cdot 3) + 4 \cdot (N \cdot 3^2) + 3^3}{N+3} = \frac{(N+3)^3 + 3N(N+3)}{N+3} = (N+3)^2 + 3N.$$

Substituting $N = 4$, the answer is $7^2 + 12 = \mathbf{61}$.

R2-3. *Let N be the number you will receive.* Compute

$$\sqrt{\frac{N^2 - 36^2 - 48^2}{37^2 - 21^2 - 28^2}}.$$

R2-3-Sol. $\boxed{\frac{11}{12}}$ The two lesser squares in the numerator and denominator combine via the 3-4-5 Pythagorean triple:

$$\begin{aligned} 36^2 + 48^2 &= (3 \cdot 12)^2 + (4 \cdot 12)^2 = (5 \cdot 12)^2 = 60^2, \\ 21^2 + 28^2 &= (3 \cdot 7)^2 + (4 \cdot 7)^2 = (5 \cdot 7)^2 = 35^2. \end{aligned}$$

The expression in the problem is equivalent to

$$\sqrt{\frac{N^2 - 36^2 - 48^2}{37^2 - 21^2 - 28^2}} = \sqrt{\frac{N^2 - 60^2}{37^2 - 35^2}} = \sqrt{\frac{N^2 - 60^2}{(37-35)(37+35)}} = \sqrt{\frac{N^2 - 60^2}{2 \cdot 72}} = \sqrt{\frac{N^2 - 60^2}{144}}.$$

Given $N = 61$, the numerator of the remaining fraction simplifies to
$61^2 - 60^2 = (61-60)(61+60) = 121$. Therefore the given expression equals $\sqrt{\frac{121}{144}} = \mathbf{\frac{11}{12}}$.

TIEBREAKER 1

A parabola $y = f(x)$ intersects the lines $y = x$ and $y = x + 4$, each exactly twice. Two of these intersection points are $(0, 0)$, and $(20, 24)$. Compute the area of the smallest possible trapezoid that has these four intersection points as vertices.

TIEBREAKER 1

A parabola $y = f(x)$ intersects the lines $y = x$ and $y = x + 4$, each exactly twice. Two of these intersection points are $(0, 0)$, and $(20, 24)$. Compute the area of the smallest possible trapezoid that has these four intersection points as vertices.

$\boxed{80}$ Denote the parabola by $y = f(x)$, and let the other intersection with the line $y = x$ be (a, a). Then the quadratic polynomial $f(x) - x$ has roots $x = 0$ and $x = a$, so express $f(x)$ as

$$f(x) = cx(x - a) + x$$

for some nonzero real constant c. This parabola passes through the point $(20, 24)$, so substitute $x = 20$ into this equation to obtain

$$24 = 20c(20 - a) + 20 \quad \Longrightarrow \quad c = \frac{1}{5(20 - a)}.$$

Therefore this quadratic equation can be expressed as

$$f(x) = \frac{1}{5(20 - a)}x(x - a) + x.$$

Find the other intersection with $y = x + 4$ in terms of a. At this intersection point (x, y), x satisfies $f(x) = x + 4$, so

$$\begin{aligned} & x + 4 &&= \tfrac{1}{5(20-a)}x(x - a) + x \\ \Longrightarrow \quad & 20(20 - a) &&= x(x - a) \\ \Longrightarrow \quad & x^2 - ax - 20(20 - a) &&= 0 \\ \Longrightarrow \quad & (x - 20)(x - a + 20) &&= 0. \end{aligned}$$

The other intersection between the parabola and $y = x + 4$ is at the point $(a - 20, a - 16)$. Therefore the problem can be solved by minimizing the area of a trapezoid with coordinates $(0, 0)$, (a, a), $(20, 24)$, and $(a - 20, a - 16)$.

The distance between the lines $y = x$ and $y = x + 4$ is $2\sqrt{2}$, so this trapezoid will have constant height as the parabola varies. The distance formula shows that the length of its lower base is $\sqrt{a^2+a^2} = |a|\sqrt{2}$, and the length of its upper base is $\sqrt{(a-20-20)^2+(a-16-24)^2} = |a-40|\sqrt{2}$. Therefore the area of this trapezoid is

$$\frac{|a|\sqrt{2}+|a-40|\sqrt{2}}{2} \cdot 2\sqrt{2} = 2(|a|+|a-40|).$$

This is minimized when a is any value between 0 and 40, and the minimum value is

$$2(a+40-a) = \mathbf{80}.$$

TIEBREAKER 2

Compute the greatest prime factor of $20! + 24!$.

TIEBREAKER 2

Compute the greatest prime factor of $20! + 24!$.

$\boxed{101}$ Notice that $20! + 24! = 20!(1 + 24 \cdot 23 \cdot 22 \cdot 21)$. Let $x = 22$. Then the given expression is equal to $1 + (x+2)(x+1)(x)(x-1) = 1 + (x^2 + 2x)(x^2 - 1)$, which equals $x^4 + 2x^3 - x^2 - 2x + 1 = (x^2 + x - 1)^2$. Substituting $x = 22$, this has the value $(484 + 22 - 1)^2 = 505^2 = 5^2 \cdot 101^2$, so the greatest prime factor of $20! + 24!$ is **101**.

TIEBREAKER 3

Steve writes all 24 permutations of the digits in 1973. Compute the sum of these 24 permutations.

TIEBREAKER 3

Steve writes all 24 permutations of the digits in 1973. Compute the sum of these 24 permutations.

$\boxed{133320}$ There will be 6 of each digit in each place (ones, tens, hundreds, and thousands). Thus the sum of the digits in the ones place is $6(1+9+7+3) = 120$. The sum of all permutations is $120(1000+100+10+1) = 120(1111) = \mathbf{133320}$.