NYSML
Team Round

20 minutes -- no calculators permitted

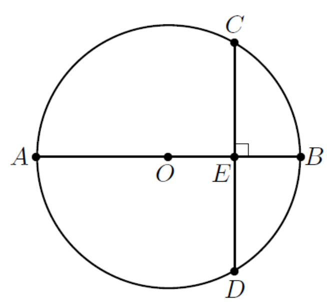
5 points each -- 50 total points

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

T1. Compute the sum of all whole numbers between 10 and 99 inclusive such that the square of the tens digit is 25 less than the number itself.

T2. A cobbler makes shoes at a constant rate, taking exactly 53 minutes to make one shoe. Over one week, she produces exactly 41 shoes. At the end of the week, she replaces one of her machines that was breaking down, and starts the second week with a new machine. She works 10 more minutes over this week than the previous week, but after counting her multiple new shoes, she determines that the new machine must be slower. Given that she still takes a whole number of minutes to make each shoe, compute the number of shoes the cobbler has made over these two weeks.

T3. A circle centered at O has diameter 50 units. Chord $\overline{CD}$ is perpendicular to diameter $\overline{AB}$ at point E.



Given that BE is an integer less than 25, compute the sum of all possible integer lengths CD.

T4. Compute the number of bases b with $10 \leq b \leq 2020$ for which $n = 2020_b$ (that is, n is the base-b number 2020) is a multiple of 10.

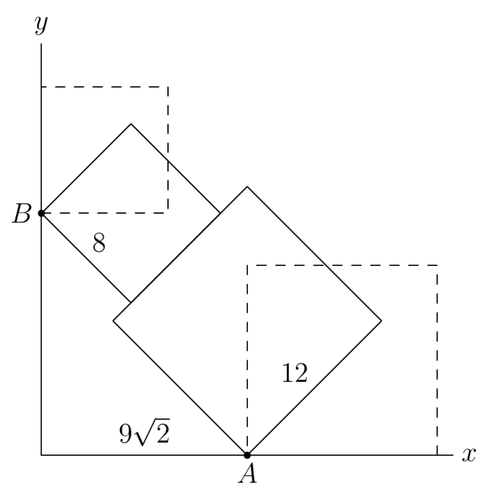
T5. Suppose that A and B are positive real numbers. Sphere 1 has surface area A square feet and volume B cubic yards. Sphere 2 has surface area B square yards and volume A cubic feet. Let r be the radius in feet of the smaller sphere. Compute $\log_3 r$. *Give your answer as a decimal.*

T6. A chess knight is placed at the origin in the xy-plane. On each turn, the knight can be moved to any lattice point that is a distance of $\sqrt{5}$ units away from its current location. Compute the least number of moves required to move the knight to the point $(2020, 2020)$.

T7. Polygon P has exactly 2020 vertices and does not intersect itself. At each vertex of P, there are two sides of measure 1 unit that are perpendicular. Compute the least possible area of polygon P in square units.

T8. Compute the coefficient of x^4 in the expansion of $(x^3 + x^2 + 2x + 3)^4$.

T9. Two squares of side lengths 12 and 8 lie in the first quadrant of the coordinate plane, with one of their sides on the positive x-axis and positive y-axis, respectively. These squares are then rotated $45°$ about their vertices A and B respectively so that the two squares touch at overlapping sides, as shown in the figure. Given that point A is $9\sqrt{2}$ units away from the origin, the distance from point B to the origin is $M\sqrt{N}$ where M and N are integers and N is not divisible by the square of any prime. Compute $M^2 + N^2$.



T10. There are many sequences of integers that can be defined using the following recursion: The first term is $a_1 = 1$, and for $n \geq 2$, $a_n = a_{n-1} + k_n$, where k_n is a randomly chosen integer in the closed interval $[1, a_{n-1}]$. Compute the expected value of a_5. *Give your answer as a decimal.*

NYSML

Team Round

20 minutes -- no calculators permitted

5 points each -- 50 total points

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

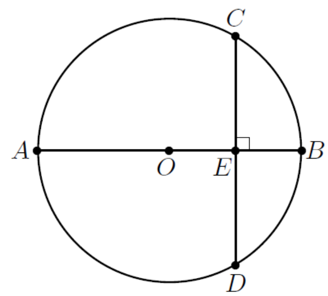
T1. Compute the sum of all whole numbers between 10 and 99 inclusive such that the square of the tens digit is 25 less than the number itself.

T1-Sol. $\boxed{\mathbf{378}}$ Write such a number in the form $N = 10T + U$ where T and U are digits and T is positive. The problem statement implies that $T^2 = 10T + U - 25$, which implies that $T^2 - 10T + (25 - U) = 0$, so $(T - 5)^2 = U$. There are several ordered pairs (T, U) of digits that satisfy this last equation: $(2, 9)$, $(3, 4)$, $(4, 1)$, $(5, 0)$, $(6, 1)$, $(7, 4)$, and $(8, 9)$. The desired sum is $29 + 34 + 41 + 50 + 61 + 74 + 89 = \mathbf{378}$.

T2. A cobbler makes shoes at a constant rate, taking exactly 53 minutes to make one shoe. Over one week, she produces exactly 41 shoes. At the end of the week, she replaces one of her machines that was breaking down, and starts the second week with a new machine. She works 10 more minutes over this week than the previous week, but after counting her multiple new shoes, she determines that the new machine must be slower. Given that she still takes a whole number of minutes to make each shoe, compute the number of shoes the cobbler has made over these two weeks.

T2-Sol. $\boxed{\mathbf{78}}$ Because the cobbler made 41 shoes at 53 minutes per shoe, she spent a total of $41 \cdot 53 = 2173$ minutes making shoes the first week. She worked 10 more minutes the second week, so her second week's work time was 2183 minutes. Factor to obtain $2183 = 37 \cdot 59$. Because her new machine is slower and because 37 and 59 are prime, she takes either 59 minutes or 2183 minutes per shoe. The latter is impossible, because she made several new shoes in the second week, so she must take 59 minutes per shoe now, and thus she made 37 shoes over the second week. Over the two weeks, she made a total of $41 + 37 = \mathbf{78}$ shoes.

T3. A circle centered at O has diameter 50 units. Chord $\overline{CD}$ is perpendicular to diameter $\overline{AB}$ at point E.



Given that BE is an integer less than 25, compute the sum of all possible integer lengths CD.

T3-Sol. $\boxed{\mathbf{132}}$ Let $BE = x$ and $CE = DE = y$. Then $AE = 50 - x$, and by Power of a Point, $y^2 = x(50 - x)$. Examining integer values from $x = 1$ to $x = 24$, the value of y is a perfect square for four values of x.
If $x = 1$, then $y = \sqrt{1 \cdot 49} = 7$ and $CD = 14$.
If $x = 5$, then $y = \sqrt{5 \cdot 45} = 15$ and $CD = 30$.
If $x = 10$, then $y = \sqrt{10 \cdot 40} = 20$ and $CD = 40$.
If $x = 18$, then $y = \sqrt{18 \cdot 32} = 24$ and $CD = 48$.
The desired sum is $14 + 30 + 40 + 48 = \mathbf{132}$.

T4. Compute the number of bases b with $10 \leq b \leq 2020$ for which $n = 2020_b$ (that is, n is the base-b number 2020) is a multiple of 10.

T4-Sol. $\boxed{\mathbf{1207}}$ Notice that $n = 2b^3 + 2b = 2b(b^2 + 1)$. Thus n is even. If n is a multiple of 10, then either $5 \mid b$ or $5 \mid (b^2 + 1)$.
There are 403 multiples of 5 in the desired interval: $10, 15, \ldots, 2020$.
Now, $b^2 + 1$ is a multiple of 5 if b^2 ends in 4 or b^2 ends in 9. There are 804 such values of b: 402 in the list $12, 17, \ldots, 2017$ and 402 in the list $13, 18, \ldots, 2018$. Thus the answer is $403 + 804 = \mathbf{1207}$.

This question is similar to Question T-1 from NYSML2010. The original question was authored by Dr. Leo Schneider, who authored NYSML from 2000 to 2010. We use the question here to honor his memory.

T5. Suppose that A and B are positive real numbers. Sphere 1 has surface area A square feet and volume B cubic yards. Sphere 2 has surface area B square yards and volume A cubic feet. Let r be the radius in feet of the smaller sphere. Compute $\log_3 r$. *Give your answer as a decimal.*

T5-Sol. $\boxed{\mathbf{1.4}}$ Let the radii of spheres 1 and 2 be r_1 feet and r_2 feet, respectively. Then Sphere 1 has surface area $4\pi r_1^2 = A$, and volume $\frac{4}{3}\pi\left(\frac{r_1}{3}\right)^3 = \frac{4\pi r_1^3}{81} = B$. Similarly, Sphere 2 has surface area $4\pi\left(\frac{r_2}{3}\right)^2 = \frac{4\pi r_2^2}{9} = B$, and volume $\frac{4}{3}\pi r_2^3 = A$. This implies the system of equations

$$A = 4\pi r_1^2 = \frac{4\pi r_2^3}{3}$$
$$B = \frac{4\pi r_1^3}{81} = \frac{4\pi r_2^2}{9}.$$

Cancelling common factors gives the simpler system $3r_1^2 = r_2^3$ and $r_1^3 = 9r_2^2$. Take the base-3 logarithm of both equations to result in a linear system:

$$1 + 2\log_3 r_1 = 3\log_3 r_2,$$
$$3\log_3 r_1 = 2 + 2\log_3 r_2.$$

The solution to this system is $(\log_3 r_1, \log_3 r_2) = \left(\frac{8}{5}, \frac{7}{5}\right)$ and, because $\log_3 x$ is an increasing function, it follows that r_2 is the smaller radius and its base-3 logarithm is $\dfrac{7}{5} = \mathbf{1.4}$.

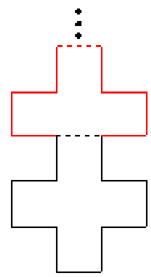
T6. A chess knight is placed at the origin in the xy-plane. On each turn, the knight can be moved to any lattice point that is a distance of $\sqrt{5}$ units away from its current location. Compute the least number of moves required to move the knight to the point $(2020, 2020)$.

T6-Sol. $\boxed{\mathbf{1348}}$ Note that any move of the knight from a point (x, y) will take that knight to either $(x \pm 1, y \pm 2)$ or $(x \pm 2, y \pm 1)$ because the only integers that satisfy $a^2 + b^2 = (\sqrt{5})^2$ are ± 1 and ± 2 in some order. Define $S = x + y$. At the beginning, $S = 0$, and when the knight reaches the point $(2020, 2020)$, $S = 4040$. Thus S must increase by 4040 for the knight to reach its final destination. However, any knight move can increase S by at most 3, and thus it must take at least $\dfrac{4040}{3} = 1346\dfrac{2}{3}$ moves to reach its final destination. This means that the minimum number of moves necessary to move the knight to $(2020, 2020)$ is at least 1347. Also note that any knight move changes the parity of S and so it is impossible to reach the point $(2020, 2020)$ in an odd number of moves. Thus at least 1348 moves are required.

A path of 1348 moves is indeed possible. First, apply the transformation $(x, y) \to (x + 2, y + 1)$ 673 times to reach $(1346, 673)$. Then apply $(x, y) \to (x + 1, y + 2)$ 673 times to obtain $(2019, 2019)$. Then move from $(2019, 2019)$ to $(2021, 2018)$ to $(2020, 2020)$. This path requires $673 + 673 + 1 + 1 = \mathbf{1348}$ moves.

T7. Polygon P has exactly 2020 vertices and does not intersect itself. At each vertex of P, there are two sides of measure 1 unit that are perpendicular. Compute the least possible area of polygon P in square units.

T7-Sol. $\boxed{\mathbf{1009}}$ Because all sides have length 1 and all pairs of adjacent sides are perpendicular, the polygon P can be placed in the coordinate plane so that all vertices are lattice points. Thus P can be dissected into unit squares. Consider how P can be built from unit squares. The first square has perimeter 4 and area 1. Each time a new unit square is added to the region, the perimeter increases by at most 2 and the area increases by 1. This implies that after a unit squares have been placed, the perimeter is less than or equal to $4 + 2(a - 1) = 2a + 2$. When the perimeter reaches 2020, the last inequality implies $2020 \le 2a + 2 \to a \ge 1009$. Equality can be obtained by laying the squares in the shape of a sequence of plus signs as shown below.



Thus the minimum possible area for P is **1009** square units.

T8. Compute the coefficient of x^4 in the expansion of $(x^3 + x^2 + 2x + 3)^4$.

T8-Sol. $\boxed{430}$ There are four ways to obtain a term with x^4. The following are the numbers of ways such a term can be obtained.

If the x^3 term is used, then one $2x$ term must be used and also two 3's. This results in $\dfrac{4!}{1!0!1!2!}(x^3)^1(x^2)^0(2x)^1(3)^2 = 216x^4$.

If the x^3 term is not used but two x^2 terms are, this requires two 3's. This results in $\dfrac{4!}{0!2!0!2!}(x^3)^0(x^2)^2(2x)^0(3)^2 = 54x^4$.

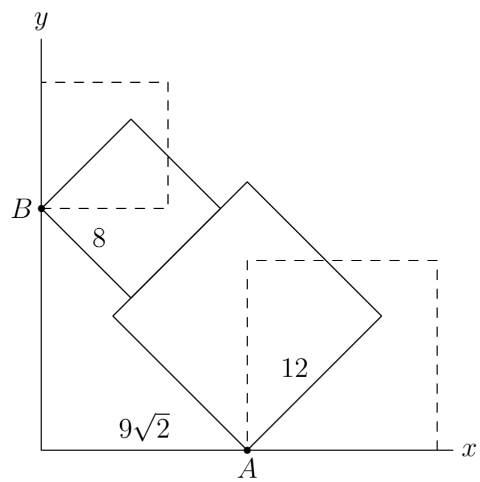
If the x^3 term is not used but one x^2 term is, then two $2x$ terms and one 3 must be used. This results in $\dfrac{4!}{0!1!2!1!}(x^3)^0(x^2)^1(2x)^2(3)^1 = 144x^4$.

If neither the x^3 nor the x^2 term is used, then four $2x$ terms must be used. This results in $\dfrac{4!}{0!0!4!0!}(x^3)^0(x^2)^0(2x)^4(3)^0 = 16x^4$.

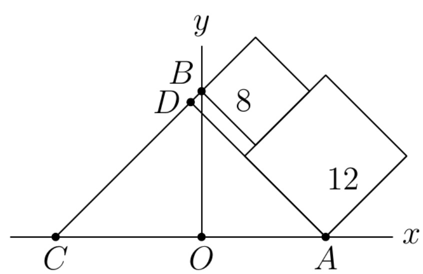
Thus the coefficient of the x^4 term is $216 + 54 + 144 + 16 = \mathbf{430}$.

This question is similar to Question T-9 from NYSML1995. Good math doesn't go bad!

T9. Two squares of side lengths 12 and 8 lie in the first quadrant of the coordinate plane, with one of their sides on the positive x-axis and positive y-axis, respectively. These squares are then rotated $45°$ about their vertices A and B respectively so that the two squares touch at overlapping sides, as shown in the figure. Given that point A is $9\sqrt{2}$ units away from the origin, the distance from point B to the origin is $M\sqrt{N}$ where M and N are integers and N is not divisible by the square of any prime. Compute $M^2 + N^2$.



T9-Sol. $\boxed{\mathbf{125}}$ Note that, because these two squares are rotated by $45°$, every line segment in the provided diagram, besides the axes, has slope 1 or -1. Let the origin be O, and extend the top-left side of the smaller square to intersect the negative x-axis at C as shown below.



Triangle $\triangle BOC$ is right and isosceles because $\overline{BC}$ has slope 1, so $BO = CO$. Let D be the foot of the perpendicular from A to line $\overleftrightarrow{BC}$; then $\triangle ACD$ is also an isosceles right triangle. Its leg $\overline{AD}$ has length $8 + 12 = 20$, so $AC = AD\sqrt{2} = 20\sqrt{2}$, but $BO = CO = AC - OA$, and it was given that $OA = 9\sqrt{2}$, so

$$BO = AC - OA = 20\sqrt{2} - 9\sqrt{2} = 11\sqrt{2}.$$

Thus the answer is $11^2 + 2^2 = \mathbf{125}$.

Note: Another way to solve this problem involves rotating the coordinate system by $45°$, so that $\overleftrightarrow{OA}$ is a line with slope -1. If the coordinates of A are $(0, 0)$, extend ray $\overrightarrow{AO}$ to the point $C = (-20, 20)$, and then note that B and C have the same y-coordinate, so once again, $OB = OC$. The rest of the solution proceeds as above.

T10. There are many sequences of integers that can be defined using the following recursion: The first term is $a_1 = 1$, and for $n \geq 2$, $a_n = a_{n-1} + k_n$, where k_n is a randomly chosen integer in the closed interval $[1, a_{n-1}]$. Compute the expected value of a_5. *Give your answer as a decimal.*

T10-Sol. $\boxed{9.125}$ Because $a_1 = 1$, there is only one choice for k_n if $n = 2$ and thus $a_2 = 1+1 = 2$. The value of a_3 is $a_2 + 1 = 3$ with probability $\frac{1}{2}$ and $a_2 + 2 = 4$ with probability $\frac{1}{2}$.

The value of a_4 is 4 only if $a_3 = 1$ and $k_4 = 1$, and this occurs with probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$.

There are two cases for which which $a_4 = 5$: either $a_3 = 3$ and $k_4 = 2$ (with probability $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$) or $a_3 = 4$ and $k_4 = 1$ (with probability $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$). Thus the probability that $a_4 = 5$ is $\frac{1}{6} + \frac{1}{8} = \frac{7}{24}$.

Using a similar approach, the following probabilities can be determined as follows.

The probability that $a_4 = 6$ is $\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{4} = \frac{7}{24}$.

The probability that $a_4 = 7$ is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

The probability that $a_4 = 8$ is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

To compute the expected value of a_5, multiply each possible value of a_5 by the probability that it occurs, and then add those products. Because there are so many possible combinations, a table might be helpful. Consider the table below, where the entries in each box show the value of a_5 that results from adding a possible value of a_4 to a value of k_5 and the probability with which that value occurs.

	$\boldsymbol{a_4 = 4}$	$\boldsymbol{a_4 = 5}$	$\boldsymbol{a_4 = 6}$	$\boldsymbol{a_4 = 7}$	$\boldsymbol{a_4 = 8}$
$\boldsymbol{k = 1}$:	$5(\frac{1}{6} \cdot \frac{1}{4})$	$6(\frac{7}{24} \cdot \frac{1}{5})$	$7(\frac{7}{24} \cdot \frac{1}{6})$	$8(\frac{1}{8} \cdot \frac{1}{7})$	$9(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 2}$:	$6(\frac{1}{6} \cdot \frac{1}{4})$	$7(\frac{7}{24} \cdot \frac{1}{5})$	$8(\frac{7}{24} \cdot \frac{1}{6})$	$9(\frac{1}{8} \cdot \frac{1}{7})$	$10(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 3}$:	$7(\frac{1}{6} \cdot \frac{1}{4})$	$8(\frac{7}{24} \cdot \frac{1}{5})$	$9(\frac{7}{24} \cdot \frac{1}{6})$	$10(\frac{1}{8} \cdot \frac{1}{7})$	$11(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 4}$:	$8(\frac{1}{6} \cdot \frac{1}{4})$	$9(\frac{7}{24} \cdot \frac{1}{5})$	$10(\frac{7}{24} \cdot \frac{1}{6})$	$11(\frac{1}{8} \cdot \frac{1}{7})$	$12(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 5}$:		$10(\frac{7}{24} \cdot \frac{1}{5})$	$11(\frac{7}{24} \cdot \frac{1}{6})$	$12(\frac{1}{8} \cdot \frac{1}{7})$	$13(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 6}$:			$12(\frac{7}{24} \cdot \frac{1}{6})$	$13(\frac{1}{8} \cdot \frac{1}{7})$	$14(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 7}$:				$14(\frac{1}{8} \cdot \frac{1}{7})$	$15(\frac{1}{8} \cdot \frac{1}{8})$
$\boldsymbol{k = 8}$:					$16(\frac{1}{8} \cdot \frac{1}{8})$

All that remains is to add the thirty products in the table. Adding them in columns, the expected value is

$$26 \cdot \frac{1}{24} + 40 \cdot \frac{7}{120} + 57 \cdot \frac{7}{144} + 77 \cdot \frac{1}{136} + 100 \cdot \frac{1}{64} = \frac{52 + 112 + 133 + 66 + 75}{48} = \frac{438}{48} = \frac{73}{8} = \mathbf{9.125}.$$

Alternate Solution: Let $E(x)$ denote the expected value of the random variable x. It is known that $E(a_1) = 1$ and $E(a_2) = 2$ because those are the only values those terms can have.
Now, use the property of additivity of expected values, which says that $E(x + y) = E(x) + E(y)$. This implies that $E(a_n) = E(a_{n-1}) + E(k_n)$. Because k_n is a randomly chosen integer between 1 and a_{n-1}, the expected value of k_n is $E(k_n) = \frac{1+a_{n-1}}{2}$. Also, on average, the expected value of a_{n-1} is a_{n-1}. Therefore $E(a_n) = E(a_{n-1}) + \frac{1+E(a_{n-1})}{2} = \frac{3}{2}E(a_{n-1}) + \frac{1}{2}$. Using this recurrence relation, the following results are obtained.
The value of $E(a_3)$ is $\frac{3}{2}E(a_2) + \frac{1}{2} = \frac{3}{2} \cdot 2 + \frac{1}{2} = \frac{7}{2}$, and this is confirmed by previous observations that $a_3 = 3$ and $a_3 = 4$ are equally likely.
The value of $E(a_4)$ is $\frac{3}{2}E(a_3) + \frac{1}{2} = \frac{3}{2} \cdot \frac{7}{2} + \frac{1}{2} = \frac{23}{4}$, which the reader is encouraged to confirm on their own.
Finally, the value of $E(a_5)$ is $\frac{3}{2}E(a_4) + \frac{1}{2} = \frac{3}{2} \cdot \frac{23}{4} + \frac{1}{2} = \frac{73}{8} = 9.125$, as above.

NYSML
MegaQuestion Round
30 minutes -- no calculators permitted
50 total points

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

THE MOD SQUAD

Background: Two integers are **equivalent mod n** if they have the same remainder when divided by n. Thus 7 and 16 are equivalent mod 3 because both have a remainder of 1 when divided by 3. Alternatively, one might note that both are 1 greater than a multiple of 3. This is written $7 \equiv 16 \pmod{3}$, and it is read "7 is equivalent to 16 mod 3."

Some people think of this as "clock arithmetic," where equivalence is taken mod 12. This is because every 12 hours, the times repeat.

A result from number theory says that if $a \equiv b \pmod{n}$, then $a^m \equiv b^m \pmod{n}$ for all positive integers m. This result may be useful during this MegaQuestion.

M1. Compute the least whole number that is equivalent to $20^{21} \pmod{21}$. This process is called **reducing** $20^{21} \pmod{21}$. *5 pts*

M2. Reduce each of the following as indicated. *6 pts*
a) Reduce $5^2 \pmod{13}$.
b) Reduce $(7 \cdot 5^2) \pmod{13}$.
c) Reduce $5^{12} \pmod{13}$.

M3. Reduce $11^{2021} \pmod{1330}$. *4 pts*

More Background: Almost every product in a grocery store comes with a Universal Product Code (UPC). This is the number that gets scanned by your supermarket cashier. Because the codes are important for a variety of reasons, there's a check system built in so the scanner can verify that the code is legitimate.

Suppose the UPC has 12 digits. Call the digits $d_1, d_2, \ldots, d_{12}$. For almost all UPCs, the value of

$$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12}$$

is equivalent to $0 \pmod{10}$. The last digit is called the **check digit** because its value is determined by the others.

Personal checks from many local banks may also use a check-digit scheme. For a nine-digit account number with digits $d_1, d_2, \ldots, d_9$, they may ensure that

$$7d_1 + 3d_2 + 9d_3 + 7d_4 + 3d_5 + 9d_6 + 7d_7 + 3d_8 + 9d_9$$

is equivalent to $0 \pmod{10}$.

M4. The UPC for Wegmans Organic Stir-Fry sauce is $0778903764X1$. Compute X. *4 pts*

M5. The UPC for a particular container of wildflower honey is $201Y30Y13039$. Compute the sum of all digits Y that satisfy the check-digit scheme. *4 pts*

M6. Compute the check digit Z for the bank code $31061483Z$. *4 pts*

M7. Suppose a bank code is 020202021. Which of the following misread codes might the computer not recognize as faulty because they satisfy the check-digit scheme? *8 pts*

A. 020222001
B. 022002021
C. 021200022
D. 020212020

Note: There may be more than one correct answer. Type all of the letters that correspond to correct answers. If you think all four letters correspond to correct answers, enter "ABCD". If you think just A and D correspond to correct answers, enter "AD".

These last few questions ask you to use modular arithmetic to solve certain problems. You can solve them any way you like, provided that you don't use any sort of computing technology.

M8. There are two distinct prime factors of 17741. Given that $135^2 \equiv 484 \pmod{17741}$, compute the positive difference of those prime factors. *5 pts*

M9. Reduce $2^{250} \pmod{143}$. *5 pts*

M10. Compute the sum of all two-digit positive integers x that satisfy $4x^2 + 9x \equiv 27 \pmod{29}$. *5 pts*

NYSML
MegaQuestion Round
30 minutes -- no calculators permitted

50 total points

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

Background: Two integers are **equivalent mod $\boldsymbol{n}$** if they have the same remainder when divided by n. Thus 7 and 16 are equivalent mod 3 because both have a remainder of 1 when divided by 3. Alternatively, one might note that both are 1 greater than a multiple of 3. This is written $7 \equiv 16 \pmod{3}$, and it is read "7 is equivalent to 16 mod 3."

Some people think of this as "clock arithmetic," where equivalence is taken mod 12. This is because every 12 hours, the times repeat.

A result from number theory says that if $a \equiv b \pmod{n}$, then $a^m \equiv b^m \pmod{n}$ for all positive integers m. This result may be useful during this MegaQuestion.

M1. Compute the least whole number that is equivalent to $20^{21} \pmod{21}$. This process is called **reducing** $20^{21} \pmod{21}$. *5 pts*

M1-Sol. $\boxed{\mathbf{20}}$ Notice that $20 \equiv (-1) \pmod{21}$, so it follows that $20^{21} \pmod{21} \equiv (-1)^{21} \pmod{21}$. Thus the answer is the least whole number equivalent to $(-1)^{21} = -1 \pmod{21}$, and that number is **20**.

M2. Reduce each of the following as indicated. *6 pts*

a) Reduce $5^2 \pmod{13}$.

b) Reduce $(7 \cdot 5^2) \pmod{13}$.

c) Reduce $5^{12} \pmod{13}$.

M2-Sol. **a)** $\boxed{\mathbf{12}}$ Notice that $5^2 = 25 = 1 \cdot 13 + 12 \equiv 12 \pmod{13}$, so the answer is **12**.

b) $\boxed{\mathbf{6}}$ It is straightforward to show that because $5^2 \equiv 12 \pmod{13}$, it follows that $(7 \cdot 5^2) \equiv (7 \cdot 12) \pmod{13}$. (One argues that **M2a** implies $5^2 - 12 = 13k$ for some integer k, so $7 \cdot (5^2 - 12) = 13 \cdot (7k)$, and thus $(7 \cdot 5^2) \equiv (7 \cdot 12) \pmod{13}$.) This is also equivalent to $(7 \cdot -1) \pmod{13} \equiv (-7) \pmod{13} \equiv 6 \pmod{13}$, so the answer is **6**.

c) $\boxed{\mathbf{1}}$ Notice that $5^{12} = (5^2)^6$, and $5^2 = 25$ is one less than 26, so $25 \equiv -1 \pmod{13}$. Thus $5^{12} \equiv (-1)^6 \equiv 1 \pmod{13}$, so the answer is **1**.

Award 2 points for each correct answer.

M3. Reduce $11^{2021} \pmod{1330}$. *4 pts*

M3-Sol. $\boxed{\mathbf{121}}$ Notice that $11^3 = 1331 \equiv 1 \pmod{1330}$, so $11^{2021} = 11^{673 \cdot 3} \cdot 11^2 \equiv 1^{673} \cdot 11^2 \equiv 11^2 = \mathbf{121} \pmod{1330}$.

More Background: Almost every product in a grocery store comes with a Universal Product Code (UPC). This is the number that gets scanned by your supermarket cashier. Because the codes are important for a variety of reasons, there's a check system built in so the scanner can verify that the code is legitimate.

Suppose the UPC has 12 digits. Call the digits $d_1, d_2, \ldots, d_{12}$. For almost all UPCs, the value of

$$3d_1 + d_2 + 3d_3 + d_4 + 3d_5 + d_6 + 3d_7 + d_8 + 3d_9 + d_{10} + 3d_{11} + d_{12}$$

is equivalent to $0 \pmod{10}$. The last digit is called the **check digit** because its value is determined by the others.

Personal checks from many local banks may also use a check-digit scheme. For a nine-digit account number with digits $d_1, d_2, \ldots, d_9$, they may ensure that

$$7d_1 + 3d_2 + 9d_3 + 7d_4 + 3d_5 + 9d_6 + 7d_7 + 3d_8 + 9d_9$$

is equivalent to $0 \pmod{10}$.

M4. The UPC for Wegmans Organic Stir-Fry sauce is $0778903764X1$. Compute X. *4 pts*

M4-Sol. $\boxed{6}$ Compute $3(0+7+9+3+6+X)+1(7+8+0+7+4+1) = 3(25+X)+1(27) = 102+3X$. For this to be a multiple of 10, it must be that $3X$ ends in 8. The only digit for which this is true is $X = \mathbf{6}$.

M5. The UPC for a particular container of wildflower honey is $201Y30Y13039$. Compute the sum of all digits Y that satisfy the check-digit scheme. *4 pts*

M5-Sol. $\boxed{7}$ Compute $3(2+1+3+Y+3+3)+1(0+Y+0+1+0+9) = 3(12+Y)+1(10+Y) = 46+4Y$. For this to be a multiple of 10, it must be that $4Y$ ends in 4. The only digits for which this is true are 1 and 6, so the answer is $1+6 = \mathbf{7}$.

M6. Compute the check digit Z for the bank code $31061483Z$. *4 pts*

M6-Sol. $\boxed{0}$ Compute $7(3+6+8)+3(1+1+3)+9(0+4+Z) = 170+9Z$. For this to be a multiple of 10, it must be that $9Z$ ends in 0. The only digit for which this is true is $Z = \mathbf{0}$.

M7. Suppose a bank code is 020202021. Which of the following misread codes might the computer not recognize as faulty because they satisfy the check-digit scheme? *8 pts*

A. 020222001
B. 022002021
C. 021200022
D. 020212020

Note: There may be more than one correct answer. Type all of the letters that correspond to correct answers. If you think all four letters correspond to correct answers, enter "ABCD". If you think just A and D correspond to correct answers, enter "AD".

M7-Sol. $\boxed{\textbf{AC}}$

A is a correct answer because the two digits that change places are both multiplied by 3 in the scheme.
B is an incorrect answer because the two digits that change places are multiplied by different numbers in the scheme.
C is a correct answer because the three digits that change places are all multiplied by 9 in the scheme.
D is an incorrect answer because the digits that change places are multiplied by different numbers in the scheme.

Award 2 points for naming each of A and C, and award 2 points for omitting each of B and D.

These last few questions ask you to use modular arithmetic to solve certain problems. You can solve them any way you like, provided that you don't use any sort of computing technology.

M8. There are two distinct prime factors of 17741. Given that $135^2 \equiv 484 \pmod{17741}$, compute the positive difference of those prime factors. *5 pts*

M8-Sol. $\boxed{\mathbf{44}}$ Notice that $135^2 \equiv 22^2 \pmod{17741}$, so it follows that $135^2 - 22^2 = (135-22)(135+22) = k \cdot 17741$ for some integer k. Arithmetic confirms that $k = 1$ and that the prime factors of 17741 are $135 - 22 = 113$ and $135 + 22 = 157$. The positive difference is $157 - 113 = \mathbf{44}$.

M9. Reduce $2^{250} \pmod{143}$. *5 pts*

M9-Sol. $\boxed{\mathbf{23}}$ Notice that $2^{10} = 1024 = 7 \cdot 143 + 23 \equiv 23 \pmod{143}$. It follows that $2^{20} \equiv 23^2 = 3 \cdot 143 + 100 \equiv 100 \pmod{143}$. This implies that $2^{50} \equiv 100 \cdot 100 \cdot 23 \pmod{143}$, and so $2^{50} \equiv 230000 = 1608 \cdot 143 + 56 \equiv 56 \pmod{143}$. Then it follows that $2^{100} \equiv 56^2 = 3136 = 21 \cdot 143 + 133 \equiv 133 \pmod{143}$. This implies that $2^{250} \equiv 133 \cdot 133 \cdot 56 = 990584 = 6927 \cdot 143 + 23 \equiv 23 \pmod{143}$. The answer is **23**.
Alternate Solution: By Euler's Theorem, it follows that $2^{\phi(143)} \equiv 1 \pmod{143}$ where $\phi(143)$ is the number of positive integers relatively prime with 143. Because $143 = 11 \cdot 13$, it follows that $\phi(143) = (11-1) \cdot (13-1) = 120$. Thus $2^{250} \equiv 2^{250 - 2 \cdot 120} \equiv 2^{10} \pmod{143}$. Reduce $2^{10} = 1024 = 7 \cdot 143 + 23 \bmod 143$ to obtain the answer of 23.

M10. Compute the sum of all two-digit positive integers x that satisfy $4x^2 + 9x \equiv 27 \pmod{29}$. *5 pts*

M10-Sol. $\boxed{\mathbf{363}}$ The given equivalence is equivalent to $4x^2 + 9x + 2 \equiv 0 \pmod{29}$. This implies that $(4x+1)(x+2) \equiv 0 \pmod{29}$. Therefore $4x + 1 \equiv 0 \pmod{29}$ or $x + 2 \equiv 0 \pmod{29}$. By inspection, $x = 7$ solves the former equivalence and $x = 27$ solves the latter. Thus the two-digit positive integers that solve the equivalence are $7 + 29 = 36$, $36 + 29 = 65$, $65 + 29 = 94$, 27, $27 + 29 = 56$, and $56 + 29 = 85$. These add to **363**.

NYSML
Individual Round
10 min/pair -- no calculators permitted

1 point each -- 150 total points

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

I1. Dr. Feng has 15 students in a statistics class. All students except Zac take an exam, and the mean score among those students is 70. Then Zac takes the exam, and the mean score of the 15 students is 72. Compute Zac's score on the exam.

I2. Compute the 2021^{st} digit of the number $N = 1223334444\ldots$, where N is created by writing each counting number k a total of k times.

I3. Let R be the set of all points in the first quadrant within 1 unit of the line $3x + 4y = 84$. Compute the area of R.

I4. A game begins with the binary sequence of eight zeros: 00000000. On each turn, a player can perform one of two actions: either change a 0 in the sequence to a 1 or reverse the sequence. Compute the number of games with exactly seven turns that result in the binary sequence 11101000.

I5. Compute the sum of all positive integers n that satisfy $(n + 1)! = 210(n - 2)!$.

I6. If every zero in the decimal expansion of

$$50! = 50 \cdot 49 \cdot 48 \cdots 2 \cdot 1$$

is replaced with the digit 2, a new number results, the sum of whose digits is 254. Given that more than half of the zeros in 50! are at the end of its expansion, compute the total number of zeros in the decimal expansion of 50!.

I7. For a particular website, passwords have eight characters, exactly three of which are distinct letters from the set $\{\mathrm{N}, \mathrm{Y}, \mathrm{S}, \mathrm{M}, \mathrm{L}\}$, and the other five of which are digits (not necessarily distinct). Compute the number of possible passwords in which exactly four of the digits are the same.

I8. Compute $\cos \frac{3\pi}{7} \cdot \cos \frac{6\pi}{7} \cdot \cos \frac{12\pi}{7}$. *Give your answer as a decimal, rounded (if necessary) to three decimal places.*

I9. Given that $f(x) = \dfrac{2}{x+5}$ for all real $x \neq -5$, compute $\left(f^{-1}(2)\right)^3$.

I10. In circle O with diameter 10, chord $\overline{AB}$ has length 5 and chord $\overline{BC}$ has length 8. The sum of both possible lengths of chord $\overline{AC}$ is $M\sqrt{N}$ where M and N are integers and N is not divisible by the square of any prime. Compute $M^2 + N^2$.

2021 Championships

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2021 Championships

I1)

I2)

Student Name:______________________ **Team Name:**_________________

2021 Championships

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2021 Championships

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I4)

Student Name:____________________ **Team Name:**________________

2021 Championships

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2021 Championships

I5)

I6)

Student Name:______________________ Team Name:__________________

2021 Championships

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2021 Championships

I7)

I8)

Student Name:______________________ Team Name:_________________

2021 Championships

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2021 Championships

I9)

I10)

Student Name:______________________ **Team Name:**_________________

NYSML
Individual Round
10 min/pair -- no calculators permitted
1 point each -- 150 total points

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I1. Dr. Feng has 15 students in a statistics class. All students except Zac take an exam, and the mean score among those students is 70. Then Zac takes the exam, and the mean score of the 15 students is 72. Compute Zac's score on the exam.

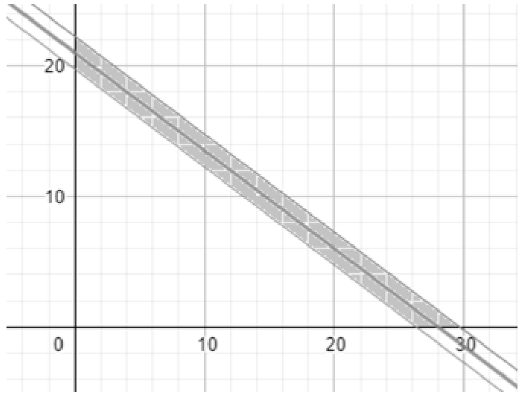
I1-Sol. $\boxed{\mathbf{100}}$ The 15 students scored a total of $15 \cdot 72 = 1080$ points. Without Zac, the other 14 students scored a total of $14 \cdot 70 = 980$ points. Therefore Zac's score was $1080 - 980 = \mathbf{100}$.

I2. Compute the 2021^{st} digit of the number $N = 1223334444\ldots$, where N is created by writing each counting number k a total of k times.

I2-Sol. $\boxed{\mathbf{5}}$ Notice that it requires $\frac{9\cdot 10}{2} = 45$ digits to write the first 9 counting numbers in the string. Then, it takes 20 digits to write 10 ten times, and 22 digits to write 11 eleven times, and so on. Thus, the key to the solution is to find the least k for which $45 + 20 + 22 + \cdots + (2k) \geq 2021$. This is equivalent to solving $\dfrac{(2k+20)(k-9)}{2} + 45 \geq 2021$, or $(k+10)(k-9) \geq 1976$. Because $54 \cdot 35 = 1890 < 1976$ and $55 \cdot 36 = 1980 > 1976$, the string in which the 2021^{st} digit occurs is a string of 45's. The $(1890+45)^{\text{th}}$ digit is the last 4 in the string of 44's, and so the 1936^{th} digit is the first 4 in the string of 45's. This implies that the 2021^{st} digit will be a $\mathbf{5}$.

I3. Let R be the set of all points in the first quadrant within 1 unit of the line $3x + 4y = 84$. Compute the area of R.

I3-Sol. $\boxed{\mathbf{70}}$ The set of points 1 unit away from the line $3x + 4y = 84$ is a set of parallel lines, so the region R is a trapezoidal region as shown.



The segment between the x- and y-intercepts of the given line is the midline of the trapezoid, whose length is the average of the bases of the trapezoid. The intercepts of the given line are $(0, 21)$ and $(28, 0)$, and so the length of the midline is $\sqrt{21^2 + 28^2} = 35$. Because the bases are 1 unit from the midline, the height of the trapezoid is 2, and so the area of the trapezoid is $A = h\left(\frac{b_1 + b_2}{2}\right) = 2 \cdot 35 = \mathbf{70}$.

I4. A game begins with the binary sequence of eight zeros: 00000000. On each turn, a player can perform one of two actions: either change a 0 in the sequence to a 1 or reverse the sequence. Compute the number of games with exactly seven turns that result in the binary sequence 11101000.

I4-Sol. $\boxed{\mathbf{840}}$ The end sequence has four 1's in it, so the sequence must have been reversed three times and 1's added four times. Accounting for the reversals, it follows that the fourth, sixth, seventh, and eighth digits of 00000000 to 1's were changed in order to get 00010111, and then that string reversed three times to obtain 11101000. Let the acts of changing these digits be called F_4, F_6, F_7, and F_8 respectively, and let the act of reversing the string be called R. Then each sequence of seven moves ending in 11101000 corresponds to an arrangement of $RRRF_4F_6F_7F_8$. Similarly, each arrangement of these letters corresponds to a sequence of seven moves. Therefore it suffices to find the number of arrangements of the letters.

The R's are indistinguishable while the F_k's are distinguishable, so there are 5 distinct letters, one of which appears three times. First consider the 7! arrangements as distinct and then correct for overcounting the 3! arrangements within the three R's, so that the number of ways of arranging these letters is

$$\frac{7!}{3!} = \frac{5040}{6} = \mathbf{840}.$$

I5. Compute the sum of all positive integers n that satisfy $(n + 1)! = 210(n - 2)!$.

I5-Sol. $\boxed{6}$ Because $(n-2)! \neq 0$, divide both sides by $(n-2)!$ to obtain

$$\frac{(n+1)(n)(n-1)(n-2)!}{(n-2)!} = 210 \to n^3 - n = 210.$$

Note that if n is of a reasonable size, $n^3 \approx 210$, so $n = 6$ is a reasonable guess for a root. Factoring yields $(n-6)(n^2+6n+35) = 0$, and the quadratic factor yields only nonreal roots, so the only solution is $n = \mathbf{6}$, which is the answer.

I6. If every zero in the decimal expansion of

$$50! = 50 \cdot 49 \cdot 48 \cdots 2 \cdot 1$$

is replaced with the digit 2, a new number results, the sum of whose digits is 254. Given that more than half of the zeros in 50! are at the end of its expansion, compute the total number of zeros in the decimal expansion of 50!.

I6-Sol. $\boxed{19}$ Use the Legendre Factorial Formula to determine the number of zeros at the end of 50!. The number of 5's in its prime factorization is

$$\left\lfloor \frac{50}{5} \right\rfloor + \left\lfloor \frac{50}{5^2} \right\rfloor = \lfloor 10 \rfloor + \lfloor 2 \rfloor = 12,$$

which is much smaller than the number of factors of 2 in the prime factorization of 50!, so there must be 12 zeros at the end of the decimal expansion of 50!. It is given that this is more than half of all the zeros in the decimal expansion of 50!, so there are at most 11 zeros among the remaining digits in 50!. In other words, there are between 12 and 23 zeros, inclusive, in 50!.
Let N be the number that results after replacing all the zeros in 50! with 2's. Recall that any number is equivalent to the sum of its digits modulo 9. It is given that the sum of the digits of N is 254, so $N \equiv 254 \equiv 2+5+4 \equiv 2 \bmod 9$. Also note that $50! \equiv 0 \bmod 9$. Therefore the additional 2's in N contribute to a total of 2 mod 9 to the digit sum of 50!. This is only possible if the number of 2's added in N is equivalent to 1 mod 9, so the number of zeros in 50! must be **19**, as this is the only number congruent to 1 mod 9 among the numbers $12, 13, \ldots, 23$.

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I7-Sol. $\boxed{1512000}$ Place the three letters first. There are eight slots, of which three will be filled by letters, and these slots can be chosen in $\binom{8}{3} = 56$ ways. For each set of chosen slots, there are $5 \cdot 4 \cdot 3 = 60$ ways to place three distinct letters in order.
Then, place the digits, four of which are the same and one of which is different. There are $10 \cdot 9 = 90$ ways to pick the two different digits, and there are 5 ways to place the "different" one in a slot, so this results in $90 \cdot 5 = 450$ possibilities.

There are therefore $56 \cdot 60 \cdot 450 = \mathbf{1512000}$ possible passwords.

I8. Compute $\cos\dfrac{3\pi}{7} \cdot \cos\dfrac{6\pi}{7} \cdot \cos\dfrac{12\pi}{7}$. *Give your answer as a decimal, rounded (if necessary) to three decimal places.*

I8-Sol. $\boxed{-0.125}$ Let $C = \cos\frac{3\pi}{7} \cdot \cos\frac{6\pi}{7} \cdot \cos\frac{12\pi}{7}$ and $S = \sin\frac{3\pi}{7} \cdot \sin\frac{6\pi}{7} \cdot \sin\frac{12\pi}{7}$. This implies that

$$C \cdot S = \frac{1}{8}\left(2\cos\frac{3\pi}{7}\sin\frac{3\pi}{7}\right)\left(2\cos\frac{6\pi}{7}\sin\frac{6\pi}{7}\right)\left(2\cos\frac{12\pi}{7}\sin\frac{12\pi}{7}\right).$$

By the double-angle identity for sines, $C \cdot S = \frac{1}{8}\sin\frac{6\pi}{7}\sin\frac{12\pi}{7}\sin\frac{24\pi}{7}$. Because $\sin\frac{24\pi}{7} = \sin\frac{10\pi}{7} = -\sin\frac{4\pi}{7} = -\sin\frac{3\pi}{7}$, this implies $C \cdot S = -\frac{1}{8}S$ or more simply,
$C = \cos\frac{3\pi}{7} \cdot \cos\frac{6\pi}{7} \cdot \cos\frac{12\pi}{7} = -\frac{1}{8} = \mathbf{-0.125}$.

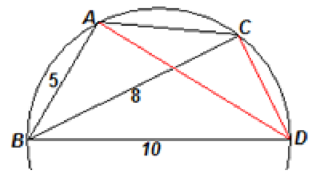
I9. Given that $f(x) = \dfrac{2}{x+5}$ for all real $x \neq -5$, compute $\left(f^{-1}(2)\right)^3$.

I9-Sol. $\boxed{-64}$ Switch the variables and solve for y in terms of x to obtain
$x = \dfrac{2}{y+5} \to xy + 5x = 2$ or $y = f^{-1}(x) = \dfrac{2-5x}{x}$. Substituting,
$\left(f^{-1}(2)\right)^3 = \left(\dfrac{2-10}{2}\right)^3 = (-4)^3 = \mathbf{-64}$.

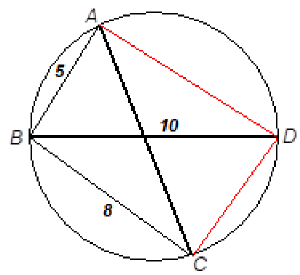
Alternate Solution: Use the fact that $f^{-1}(b) = a$ if and only if $f(a) = b$. Seek an x such that $\dfrac{2}{x+5} = 2$. By inspection, if $x = -4$, then the denominator is 1 and the fraction has value 2. Thus $f^{-1}(2) = -4$, and the result follows similarly.

I10. In circle O with diameter 10, chord $\overline{AB}$ has length 5 and chord $\overline{BC}$ has length 8. The sum of both possible lengths of chord $\overline{AC}$ is $M\sqrt{N}$ where M and N are integers and N is not divisible by the square of any prime. Compute $M^2 + N^2$.

I10-Sol. $\boxed{73}$ Let D be the point on the circle such that $\overline{BD}$ is a diameter. Then triangles ABD and BCD are right triangles. By the Pythagorean Theorem, $AD = \sqrt{10^2 - 5^2} = 5\sqrt{3}$ and $CD = \sqrt{10^2 - 8^2} = 6$. There are two cases, each of which results in a possible length of $\overline{AC}$.
Case 1: Suppose A and C lie on the same side of $\overline{BD}$. Then because $ABDC$ is inscribed in a circle, Ptolemy's Theorem applies.



By Ptolemy's Theorem, $AC \cdot 10 + 6 \cdot 5 = 8 \cdot 5\sqrt{3}$ and thus $AC = 4\sqrt{3} - 3$.
Case 2: Suppose A and C lie on opposite sides of $\overline{BD}$. Then because $ABCD$ is inscribed in a circle, Ptolemy's Theorem applies.



By Ptolemy's Theorem, $5\sqrt{3} \cdot 8 + 5 \cdot 6 = 10 \cdot AC$ and thus $AC = 4\sqrt{3} + 3$.
The sum of the two possible lengths AC is $8\sqrt{3}$, so the answer is $8^2 + 3^2 = \mathbf{73}$.

NYSML

Relay Round

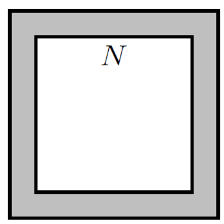
no calculators

Each Round: sub-teams of 3 -- 5 points at 3 minutes, 3 points at 6 minutes

2021 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. The solutions of the equation $2x^2 + 6x + 1 = 0$ are $r_1 = \frac{-3-\sqrt{7}}{2}$ and $r_2 = \frac{-3+\sqrt{7}}{2}$. Compute $\dfrac{r_1}{r_2} + \dfrac{r_2}{r_1}$.

R1-2. *Let N be the number you will receive.* A region bounded by two concentric squares with vertical and horizontal sides has an area of A square units. The sum of the perimeters of the two concentric squares is P units. The inner square has a side of length N and the bounded region has uniform width.



Given that $A = P$, compute the length of the side of the square with area A.

R1-3. *Let N be the number you will receive.* Richard is riding his bicycle at a constant rate of N miles per hour from his house to Jane's house. If Richard had increased his rate by 1 mile per hour, he would have arrived at Jane's house on time. If instead he had left one minute earlier, he would have also arrived at Jane's house on time. Compute the distance from Richard's house to Jane's house in miles. *Give your answer as a decimal.*

R2-1. Of the six interior angles in a hexagon, four measure $150°$ and the other two are congruent acute angles. Compute the degree measure of one of these congruent acute angles. Pass back the number without the degree symbol.

R2-2. *Let N be the number you will receive.* For some positive integers a, b, and c, the equation

$$x^4 - ax^3 + bx^2 - cx + 2N + 10 = 0$$

has four positive integer roots. One of the roots is 1 and the other three roots are distinct primes. Compute a.

R2-3. *Let N be the number you will receive.* Compute the number of ordered pairs whose elements are chosen from $\{1, 2, 3, \ldots, 15\}$, and whose sum is N.

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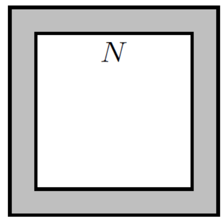

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NYSML

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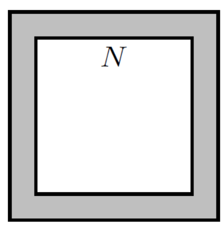
2021 Championships *The word "compute" calls for an exact answer in simplest form.*

R1-1. The solutions of the equation $2x^2 + 6x + 1 = 0$ are $r_1 = \frac{-3-\sqrt{7}}{2}$ and $r_2 = \frac{-3+\sqrt{7}}{2}$. Compute $\dfrac{r_1}{r_2} + \dfrac{r_2}{r_1}$.

R1-1-Sol. $\boxed{\mathbf{16}}$ Rewrite $\dfrac{r_1}{r_2} + \dfrac{r_2}{r_1}$ as $\dfrac{r_1^2 + r_2^2}{r_1 r_2} = \dfrac{(r_1+r_2)^2 - 2r_1r_2}{r_1r_2} = \dfrac{(r_1+r_2)^2}{r_1r_2} - 2$. Using Vieta's formulas, $r_1 + r_2 = -6/2 = -3$ and $r_1r_2 = 1/2$, so the answer is $\dfrac{(-3)^2}{1/2} - 2 = \mathbf{16}$.

R1-2. *Let N be the number you will receive.* A region bounded by two concentric squares with vertical and horizontal sides has an area of A square units. The sum of the perimeters of the two concentric squares is P units. The inner square has a side of length N and the bounded region has uniform width.



Given that $A = P$, compute the length of the side of the square with area A.

R1-2-Sol. $\boxed{\mathbf{12}}$ Let w denote the uniform width of the bounded region. Then $A = P \rightarrow (N+2w)^2 - N^2 = 4(N+2w) + 4N \rightarrow 4Nw + 4w^2 = 8(N+w)$, so $4w(N+w) = 8(N+w)$ and $4w = 8 \rightarrow w = 2$. Then $A = 8N + 16$, and substituting gives an answer of $\sqrt{8(16)+16} = \sqrt{9(16)} = 3(4) = \mathbf{12}$.

R1-3. *Let N be the number you will receive.* Richard is riding his bicycle at a constant rate of N miles per hour from his house to Jane's house. If Richard had increased his rate by 1 mile per hour, he would have arrived at Jane's house on time. If instead he had left one minute earlier, he would have also arrived at Jane's house on time. Compute the distance from Richard's house to Jane's house in miles. *Give your answer as a decimal.*

R1-3-Sol. $\boxed{\mathbf{2.6}}$ Let t be the time it takes to bike from Richard's house to Jane's house and let d be the desired distance. The problem statement implies $N\left(t + \dfrac{1}{60}\right) = (N+1)t$, and this implies $t = \dfrac{N}{60}$. Thus $d = \dfrac{N}{60}(N+1)$. Substituting, $d = \dfrac{12}{60} \cdot 13 = \dfrac{13}{5} = \mathbf{2.6}$ miles.

R2-1. Of the six interior angles in a hexagon, four measure 150° and the other two are congruent acute angles. Compute the degree measure of one of these congruent acute angles. Pass back the number without the degree symbol.

R2-1-Sol. $\boxed{\mathbf{60}}$ The sum of the degree measures of the interior angles of a hexagon is $(6-2)(180^\circ) = 720^\circ$. Subtracting, the other two angles have degree measures that add to $720^\circ - 4(150^\circ) = 120^\circ$, so each acute angle measures $120^\circ \div 2 = 60^\circ$. Pass back **60**.

R2-2. *Let N be the number you will receive.* For some positive integers a, b, and c, the equation

$$x^4 - ax^3 + bx^2 - cx + 2N + 10 = 0$$

has four positive integer roots. One of the roots is 1 and the other three roots are distinct primes. Compute a.

R2-2-Sol. $\boxed{\mathbf{21}}$ Call the roots 1, x_1, x_2, and x_3. Then the product of the roots is $2N + 10 = 2(N + 5) = x_1x_2x_3$. Assuming N is an integer, then without loss of generality, let $x_3 = 2$. This implies $x_1x_2 = N + 5$. Substituting, $x_1x_2 = 65 = 5(13)$. The value of a is the sum of the roots, or $1 + 2 + 5 + 13 = \mathbf{21}$.

R2-3. *Let N be the number you will receive.* Compute the number of ordered pairs whose elements are chosen from $\{1, 2, 3, \ldots, 15\}$, and whose sum is N.

R2-3-Sol. $\boxed{\mathbf{10}}$ If $N \geq 30$, the answer is 0. If $0 \leq N \leq 29$, the desired ordered pairs are of the form $(N - 15 + k, 15 - k)$ for $0 \leq k \leq 30 - N$, and there are a total of $30 - N + 1$ such pairs. Substituting, there are $30 - 21 + 1 = \mathbf{10}$ possible ordered pairs. These are $(6, 15), (7, 14), \ldots, (15, 6)$.

Note: This relay is similar to a relay from NYSML2005. The original relay was authored by Dr. Leo Schneider, who authored NYSML from 2000 to 2010. We include this relay to honor his memory.

TIEBREAKER 1

Triangle ABC has integer side lengths and is not isosceles. Let the angle bisector from point B intersect $\overline{AC}$ at point D. Given that $AD = 5$ and $BC = 9$, compute $[ABD]^2$, the square of the area of $\triangle ABD$.

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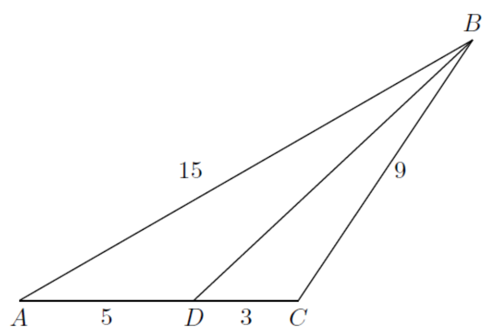
TB1-Ans: $\boxed{\mathbf{350}}$

TB1-Sol: The Angle Bisector Theorem implies that $\frac{AB}{AD} = \frac{BC}{CD}$, which implies $(AB)(CD) = 5 \cdot 9 = 45$. Because AB and CD are both integers, because of the Triangle Inequality, and because $\triangle ABC$ is not isosceles, it follows that $AB = 15$ and $CD = 3$. Consider the following diagram.



Because triangles ABD and BDC share a common altitude to bases $\overline{AD}$ and $\overline{DC}$, respectively, the ratio of their areas equals the ratio of AD to DC, which is $\frac{5}{3}$. So to find $[ABD]$, find $[ABC]$ and multiply by $\frac{5}{8}$.

By Heron's formula,

$[ABC] = \sqrt{16(16-9)(16-8)(16-15)} = 8\sqrt{14}$ and thus $[ABD] = 5\sqrt{14}$. The answer is $25 \cdot 14 = \mathbf{350}$.

TIEBREAKER 2

Compute the least positive integer with 7 digits and exactly 7 positive divisors.

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TB2-Ans: $\boxed{\mathbf{1771561}}$

TB2-Sol: If an integer k can be factored into primes p_i as $p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_n^{e_n}$, then the number of positive integer divisors that k has is given by $(e_1+1)(e_2+1)(e_3+1)\cdots(e_n+1)$. Because the problem calls for an integer with 7 divisors and 7 is prime, the above product of binomials can only consist of one parenthesized binomial, namely, (e_1+1), and so $e_1+1=7 \to e_1=6$. Thus the desired integer is of the form p^6 where p is a prime. Because p^6 must have 7 digits, it follows that $p \geq 10$. The least prime p with $p \geq 10$ is 11, and so the answer is $11^6 = (11^3)^2 = 1331^2 = \mathbf{1771561}$.

TIEBREAKER 3

Given that a_i, a_j, and a_k are positive integers between 1 and 9, inclusive such that $a_i a_j a_k = 10(a_i + a_j + a_k)$, compute $a_i + a_j + a_k$.

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Given that a_i, a_j, and a_k are positive integers between 1 and 9, inclusive such that $a_i a_j a_k = 10(a_i + a_j + a_k)$, compute $a_i + a_j + a_k$.

TB3-Ans: $\boxed{\mathbf{18}}$

TB3-Sol: Because 10 divides the product of three digits, at least one of the digits is 5; suppose that $a_i = 5$ without loss of generality. This implies $a_j a_k = 10 + 2a_j + 2a_k$. Add 4 to both sides, factor, and rearrange terms to obtain $(a_j - 2)(a_k - 2) = 14$. Because the variables are digits, the only solutions to this last equation are that a_j and a_k are 4 and 9 in some order. Thus
$a_i + a_j + a_k = 4 + 5 + 9 = \mathbf{18}$.